MATH 0290 - Practice Final exam

Name: ______________________________

Read all of the following information before starting the exam:

- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Circle or otherwise indicate your final answers.
- This practice final has 15 problems. The actual final has 10 problems.
- Calculators are neither required nor permitted.
- Good luck!

1. (*10 points*)

Solve the following initial value problems:

(a) $y' = \frac{te^t}{y}$, $\qquad y(1) = 1.$

(b) $y' = y^2 x$, $\qquad y(3) = 1.$

2. (*10 points*)

Solve the following differential equations. If an explicit solution cannot be found, leave the solution in an implicit form:

(a)

$$e^t y y' = e^{-y} + e^{-2t-y}$$

(b)

$$y' + 3t^2 y = t^2$$

3. (*10 points*)

(a) Show that the transformation $y(x) = 1/z(x)$ transforms the nonlinear ODE

$$xy' + y = y^2$$

to the linear ODE

$$z' - \frac{z}{x} = -\frac{1}{x}$$

(b) Give the general solution of $y(x)$. Plot the specific solutions for initial conditions $y(1) = 1$ and $y(1) = 2$ for $0 < x < 2$.

4. (*10 points*)

Use the method of variation of parameters OR the method of undetermined coefficients to find a particular solution, $y_p(t)$, to the following differential equation:

$$y'' - 2y' + y = e^{-t}\cos t.$$

5. (*10 points*)

Solve the following initial value problems:

(a)

$$y'' + y' - 6y = 2\sin(2t); \qquad y(0) = 0, y'(0) = 0;$$

(b)

$$y'' + y' - 6y = e^{-3t}; \qquad y(0) = 0, y'(0) = 0;$$

6. (*10 points*)

My apartment has an ant infestation. The growth rate of the ants is r and the population size is currently estimated to be A_0. I have decided to fumigate my apartment with a fumigation agent that kills F ants per hour. The governing ODE for the ant population $A(t)$ is $A' = rA - F$ with $A(0) = A_0$. What is the minimal strength fumigation agent (i.e the smallest F) that I can use to ensure that after T hours of fumigation all of the ants are dead? Write your solution in terms of r, T, and A_0.

7. (*10 points*)

Consider the system $\mathbf{y}' = A\mathbf{y}$ where:

$$A = \begin{pmatrix} -3 & -4 \\ -3 & 1 \end{pmatrix}.$$

(a) Find the fundamental solution set for the above system.

(b) Plot the phase portrait of the solution near $\mathbf{y} = (0,0)^T$.

8. (*10 points*)

Consider the system $\mathbf{y}' = A\mathbf{y}$ where

$$A = \begin{pmatrix} -1 & 3 \\ -3 & -1 \end{pmatrix}.$$

Find the solution $\mathbf{y}(t)$ with initial condition $\mathbf{y}(0) = (1, 0)^T$. Express your solution as a real vector in $\mathbb{R}^2$ (i.e no complex numbers).

9. (*10 points*)

Consider the following nonlinear system:

$$\begin{aligned} x' &= 2(xy-1) \\ y' &= x - y^3. \end{aligned}$$

(a) Find all equilibrium points (hint: there is more than one).

(b) Classify the stability of all equilibrium points.

10. (*10 points*)

Consider the following model of competition between two animal species:

$$\begin{aligned} X' &= (1 - X - Y) \qquad (X \geq 0) \\ Y' &= (1 - 2Y - X/2) \qquad (Y \geq 0) \end{aligned}$$

(a) Find all equilibrium points, and classify the stability of each equilibrium point.

(b) Draw the qualitative phase portrait of the system.

(c) Are there initial conditions which result in one of the populations going extinct?

11. (*10 points*)

Using the Laplace method solve the following initial value problem:

$$y'' + 2y' - 3y = 16e^{-3t}; \qquad y(0) = 1, y'(0) = -3.$$

12. (*10 points*)

Solve the following initial value problem:

$$y' + y = g(t); \qquad y(0) = 0,$$

with

$$g(t) = \begin{cases} t^2 & \text{if } 0 \le t < 2 \\ 4 & \text{if } t \ge 2. \end{cases}$$

13. (*10 points*)

The convolution of two functions g and h is defined as $g * h(t) = \int_0^t g(u)h(t-u)du$. Let $f(t) = e^{-2t}$.

(a) Evaluate $f * f$.

(b) Use your answer from (a) and verify that:

$$\mathcal{L}(f * f) = (\mathcal{L}(f))^2.$$

(c) Solve the following initial value problem:

$$y' + y = f * f, \qquad y(0) = 0.$$

14. (*10 points*)

Write $f(x)$ as a Fourier series over $-1 \le x \le 1$ where

$$f(x) = \begin{cases} 1 + 2x & \text{if } -1 \le x < 0 \\ 1 - 2x & \text{if } 0 \le x \le 1. \end{cases}$$

15. (*10 points*)

Write $f(x)$ as a Fourier series over $-2 \leq x \leq 2$ where

$$f(x) = \begin{cases} -1 & \text{if } -2 \leq x < 0 \\ 1 & \text{if } 0 \leq x \leq 2. \end{cases}$$

Table of Laplace Transforms

$f(t)$	$\mathcal{L}(f(t)) = F(s)$	condition
1	$\frac{1}{s}$	$s > 0$
t^n	$\frac{n!}{s^{n+1}}$	$s > 0$
$\sin(at)$	$\frac{a}{s^2+a^2}$	$s > 0$
$\cos(at)$	$\frac{s}{s^2+a^2}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	$s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$
$\delta(t-c)$	e^{-cs}	$s > 0$
$H(t-c)$	$\frac{e^{-cs}}{s}$	$s > 0$

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