MATH 0290 - Differential Equations
SAMPLE FINAL EXAM

Answer Key

1. Solve the initial-value problem. Show all the work. Mention a type of the given differential equation.

(a) (5 points) $y' = 2xe^{x^2-y}$, $y(0) = 1$.

Separable. $\frac{dy}{dx} = 2xe^{x^2}e^{-y}$

$\int e^y dy = \int 2xe^{x^2}dx$, $e^y = e^{x^2} + C$

$y(0)=1$, $e^1 = e^0 + C$, $e = 1 + C$, $C = e-1$

$e^y = e^{x^2} + e - 1$

$y = \ln(e^{x^2} + e - 1)$

Note: $y = \ln|e^{x^2}+e-1|$ is not correct answer

(b) (5 points) $t^2y' - ty + 2 = 0$, $y(1) = 0$.

$y' - t^{-1}y = -2t^{-2}$ First order, linear, inhomogeneous

$u(t) = e^{-\int t^{-1}dt} = e^{-\ln t} = e^{\ln t^{-1}} = t^{-1}$

$t^{-1}y' - t^{-2}y = -2t^{-3}$

$(t^{-1}y)' = -2t^{-3}$, $t^{-1}y = t^{-2} + C$

$y(1) = 0 \Rightarrow 0 = 1 + C$, $C = -1$

$t^{-1}y = t^{-2} - 1$, $\boxed{y = t^{-1} - t}$

2. Find the general solution of the given differential equation.

(a) (5 points) $y'' + 4y = 10 \sin 3t$

Char. eq $\lambda^2+4=0$, $\lambda = 2i$, $\bar{\lambda} = -2i$

FSS: $y_1 = \cos 2t$, $y_2 = \sin 2t$

The right hand side is not similar to y_1 or y_2

Then $y_p = a \sin 3t + b \cos 3t$, $y_p'' = -9 y_p$

$y_p'' + 4y_p = -9y_p + 4y_p = -5y_p = -5a \sin 3t - 5b \cos 3t$
$= 10 \sin 3t$

$a = -2$, $y_p = -2 \sin 3t$

Gen. sln: $y(t) = c_1 \cos 2t + c_2 \sin 2t - 2 \sin 3t$

(b) (5 points) $y'' + 4y = 4 \sec 2t$

From the previous problem FSS $y_1 = \cos 2t$
$y_2 = \sin 2t$

$y_p = v_1 y_1 + v_2 y_2$ where

$v_1 = -\int \frac{y_2 g}{W} dt$, $v_2 = \int \frac{y_1 g}{W} dt$, $W = \begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix} = 2$

$v_1 = -2\int \sin 2t \sec 2t \, dt = -2\int \frac{\sin 2t}{\cos 2t} dt = \ln|\cos 2t|$

use $u = \cos 2t$
$du = -2 \sin 2t \, dt$

$v_2 = 2\int \cos 2t \sec 2t \, dt = 2\int dt = 2t$

Gen sln: $y(t) = c_1 \cos 2t + c_2 \sin 2t + \cos 2t \ln|\cos 2t|$
$+ 2t \sin 2t$

3. (5 points) A 0.3 kg mass is attached to a spring that has a spring constant 30 kg/s^2. The system is displaced 2 m from its equilibrium position and released from rest. If there is no damping, find the amplitude, frequency, and phase of the resulting motion.

Equation: $0.3x'' + 30x = 0$, ICs: $x(0) = 2$, $x'(0) = 0$

$x'' + 100x = 0$, $\lambda^2 + 100 = 0$, $\lambda = 10i$

$x(t) = C_1 \cos 10t + C_2 \sin 10t$ $\quad x(0) = C_1 = 2$

$x(t) = 2\cos 10t + C_2 \sin 10t$, $x'(t) = -20\sin 10t + 10C_2 \cos 10t$

$x'(0) = 10C_2 = 0 \Rightarrow C_2 = 0 \Rightarrow x(t) = 2\cos 10t$

$A = 2$, $\omega = 10$, $\varphi = 0$

4. (5 points) Use the Laplace transform to solve the initial-value problem

$$3y' + 5y = -2e^{-t}, \qquad y(0) = -2$$

$L[3y' + 5y] = L[-2e^{-t}]$, $3sY - 3y(0) + 5Y = -\frac{2}{s+1}$

$(3s+5)Y = -\frac{2}{s+1} - 6$, $Y = -\frac{2}{(s+1)(3s+5)} - \frac{6}{3s+5}$

$Y = -\frac{1}{s+1} + \frac{3}{3s+5} - \frac{6}{3s+5} = -\frac{1}{s+1} - \frac{3}{3s+5}$

$y = L^{-1}[Y] = -L^{-1}\left[\frac{1}{s+1}\right] - 3L^{-1}\left[\frac{1}{3s+5}\right]$

$L^{-1}\left[\frac{1}{s+1}\right] = e^{-t}$, $L^{-1}\left[\frac{1}{3s+5}\right] = L^{-1}\left[\frac{1/3}{s+5/3}\right]$

$= \frac{1}{3}e^{-\frac{5}{3}t}$

$y(t) = -e^{-t} - e^{-\frac{5}{3}t}$

5. (5 points) Find the inverse Laplace transform of the function $F(s) = \frac{se^{-s}}{s^2+9}$.

Create the piecewise definition of your solution that does not use the Heaviside function.

$$L^{-1}\left[\frac{se^{-s}}{s^2+9}\right](t) = L^{-1}\left[e^{-s}\frac{s}{s^2+3^2}\right](t) = H(t-1)L^{-1}\left[\frac{s}{s^2+3^2}\right](t-1)$$

$$L^{-1}\left[\frac{s}{s^2+3^2}\right](t) = \cos 3t$$

Hence, $L^{-1}\left[\frac{se^{-s}}{s^2+9}\right](t) = H(t-1)\cos(3(t-1))$

6. (5 points) Find the unit impulse response to the initial-value problem

$$y'' - 6y' + 9y = \delta(t), \qquad y(0) = y'(0) = 0$$

Char. polynomial: $s^2 - 6s + 9 = (s-3)^2$

$$e(t) = L^{-1}\left[\frac{1}{(s-3)^2}\right](t) = te^{3t}$$

7. (5 points) For the initial-value problem $y' = ty$, $y(0) = 5$ calculate the first two iterations y_1 and y_2 of Euler's method if the step size is $h = 0.1$.

$t_0 = 0, \quad t_1 = 0.1, \quad y_0 = 5$

EM: $y_{n+1} = y_n + h \cdot t_n y_n = (1 + h t_n) y_n$

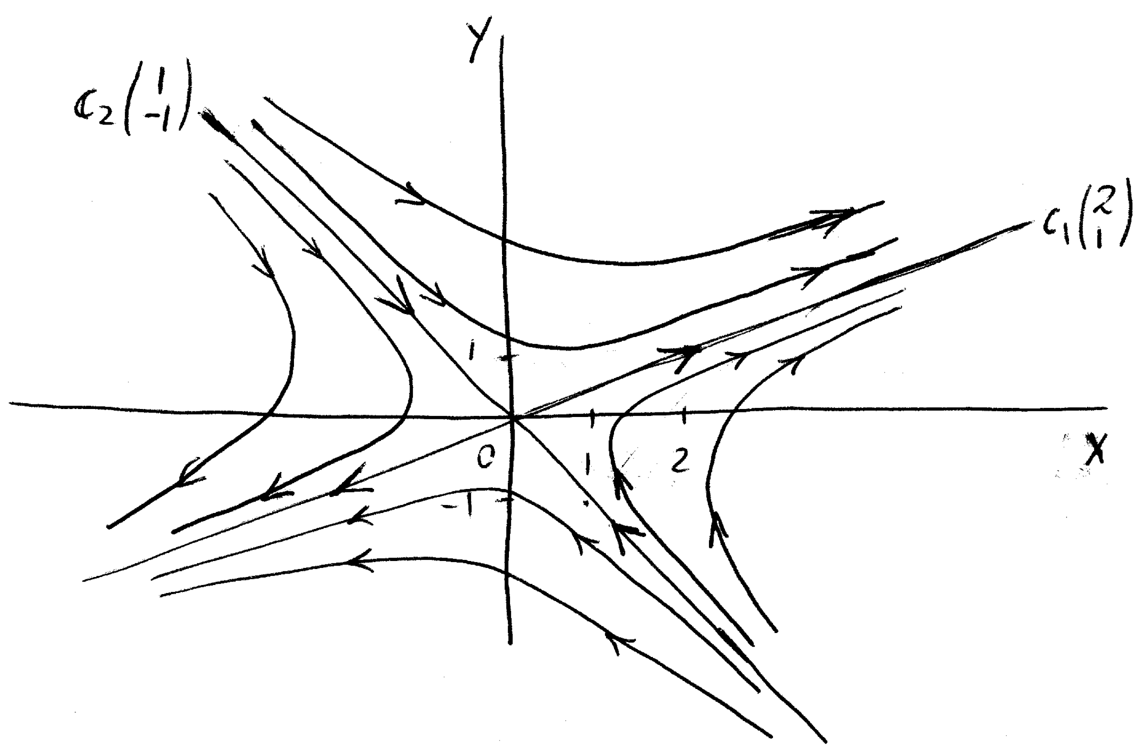
$y_1 = (1 + 0.1 \times 0) \times 5 = 1 \cdot 5 = 5$

$y_2 = (1 + (0.1)(0.1)) \times 5 = 1.01 \times 5 = 5.05$

8. (10 points) Let $\bar{y}(t) = C_1 e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ be a general solution of the system $\bar{y}' = A\bar{y}$
for some matrix A.

Scketch a phase plane portrait of the system near the origin. Use arrows to indicate the direction of motion on all solution curves. Name the type of the equilibrium point (a saddle, a nodal sink, a nodal source, a center, a spiral sink, or a spiral source).

$\lambda_1 = -2, \ \lambda_2 = 1$. The EP $(0,0)$ is a saddle.



10. For the system of differential equations

$x' = x^2 - y^2$

$y' = (y-2)(y-x^2)$

(a) (3 points) find x-nullclines and y-nullclines.

X-nullcline: $x^2-y^2=0$, $(x-y)(x+y)=0$,

X-nullcl is a union of lines $y=-x$ and $y=x$

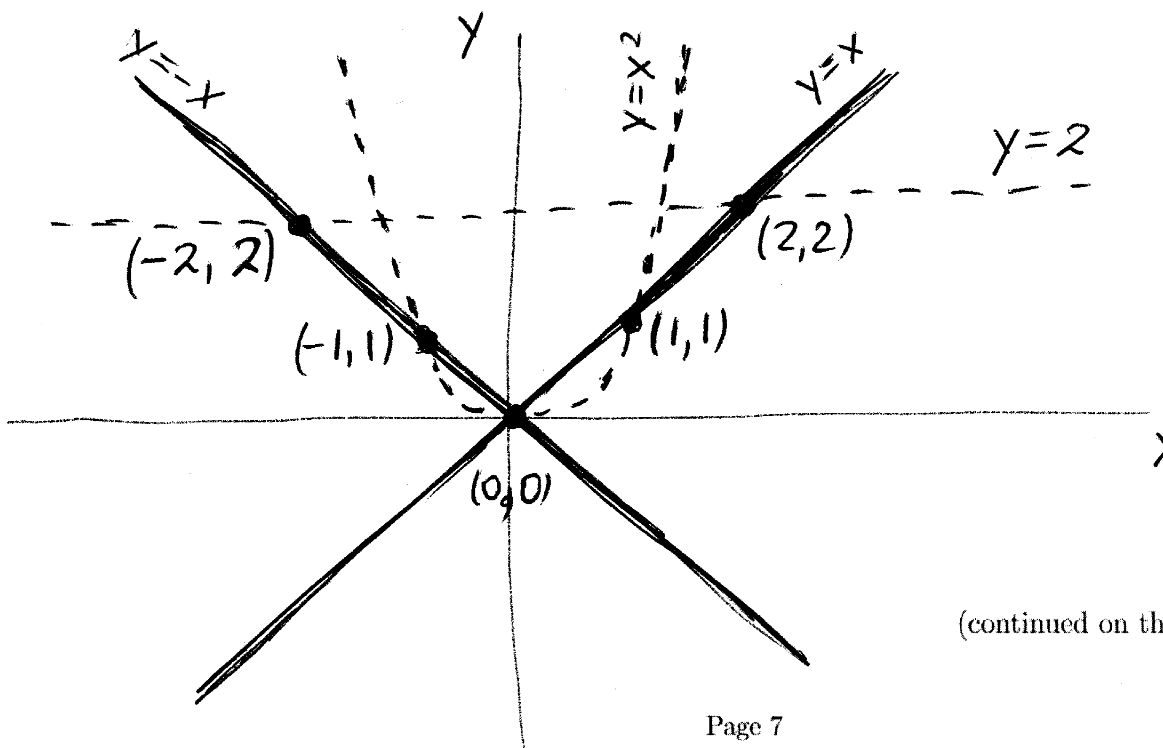
y-nullcl: $(y-2)(y-x^2)=0$ is a union of the line $y=2$ and the parabola $y=x^2$

(b) (3 points) calculate the coordinates of the equilibrium points.

EPs: $(0,0)$, $(-2,2)$, $(2,2)$, $(-1,1)$, $(1,1)$

(c) (3 points) Plot x-nullclines and y-nullclines. Use solid lines for x-nullclines and dashed lines

for y-nullclines. Plot the equilibrium points and label them with their coordinates.



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(d) (3 points) Use Jacobian to classify the equilibrium point $(-2, 2)$ (a saddle, a nodal source, etc.)

$$J = \begin{bmatrix} 2x & -2y \\ -2x(y-2) & y-x^2+y-2 \end{bmatrix}$$

$$J(-2,2) = \begin{bmatrix} -4 & -4 \\ 0 & -2 \end{bmatrix} \qquad \begin{array}{l} T = -6 < 0 \\ D = 8 > 0 \end{array}$$

$$T^2 - 4D = 36 - 32 = 4 > 0$$

The EP is a nodal sink

(e) (3 points) Use Jacobian to determine if the equilibrium point $(-2, 2)$ is stable or unstable.

$$\lambda^2 - T\lambda + D = 0$$

$$\lambda^2 + 6\lambda + 8 = 0$$

$$\lambda = \frac{1}{2}\left[-6 \pm \sqrt{36-32}\right] = -3 \pm 1$$

$\lambda_1 = -4, \ \lambda_2 = -2$ Both e-values are negative

Hence, the EP is asymptotically stable

11. (10 points) Expand the function $f(x) = x$ in a Fourier sine series valid on the interval $0 \le x \le 2\pi$.

$$x = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{2\pi} x\right) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n}{2} x\right)$$

where $$b_n = \frac{2}{2\pi} \int_0^{2\pi} x \sin\left(\frac{n\pi}{2\pi} x\right) dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} x \sin\left(\frac{n}{2} x\right) dx$$

by parts: $u = x \qquad dv = \sin\left(\frac{n}{2}x\right)dx$

$du = dx \qquad v = -\frac{2}{n}\cos\left(\frac{n}{2}x\right)$

$$b_n = \frac{1}{\pi}\left[-\frac{2}{n} x \cos\left(\frac{n}{2}x\right)\Big|_0^{2\pi} + \frac{2}{n}\int_0^{2\pi} \cos\left(\frac{n}{2}x\right)dx\right]$$

$$= \frac{1}{\pi}\left[-\frac{4\pi}{n}\cos(n\pi) + \underbrace{\frac{4}{n^2}\sin\left(\frac{n}{2}x\right)\Big|_0^{2\pi}}_{=0}\right]$$

$$= -\frac{1}{\pi}\left[\frac{4\pi}{n}(-1)^n\right] = \frac{4}{n}(-1)^{n+1}$$

Therefore, $$x = 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n}{2}x\right)$$

12. (10 points) Find the temperature $u(t, x)$ in a rod modeled by the initial/boundary value problem

$$u_t = u_{xx}, \quad \text{for } t > 0, \; 0 < x < 2\pi,$$
$$u(0, t) = 0, \quad u(2\pi, t) = 0, \quad \text{for } t > 0,$$
$$u(x, 0) = \frac{x}{4}, \quad \text{for } 0 \le x \le 2\pi.$$

You may use results obtained in the previous problem.

$K = 1, \; L = 2\pi, \; T_0 = T_L = 0 \Rightarrow u_s(x) = 0$

Then $u(x,t) = v(x,t) = \sum_{n=1}^{\infty} b_n e^{-\frac{n^2}{4}t} \sin(\frac{n}{2}x)$

where $b_n = \frac{2}{2\pi} \int_0^{2\pi} \frac{x}{4} \sin(\frac{n\pi}{2\pi}x)dx$

$b_n = \frac{1}{4} \cdot \frac{1}{\pi} \int_0^{2\pi} x \sin(\frac{n}{2}x)dx$

From the previous problem

$b_n = \frac{1}{4} \times \frac{4}{n}(-1)^{n+1} = \frac{(-1)^{n+1}}{n}$

Therefore,

$$u(x,t) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^{-\frac{n^2}{4}t} \sin(\frac{n}{2}x)$$