MATH 0290 - Differential Equations
SAMPLE FINAL EXAM

1. Solve the initial-value problem. Show all the work. Mention a type of the given differential equation.

   (a) (5 points) $y' = 2xe^{x^2-y}, \qquad y(0) = 1.$

   (b) (5 points) $t^2y' - ty + 2 = 0, \qquad y(1) = 0.$

2. Find the general solution of the given differential equation.

(a) (5 points) $y'' + 4y = 10\sin 3t$

(b) (5 points) $y'' + 4y = 4\sec 2t$

3. (5 points) A 0.3 kg mass is attached to a spring that has a spring constant 30 kg/s^2. The system is displaced 2 m from its equilibrium position and released from rest. If there is no damping, find the amplitude, frequency, and phase of the resulting motion.

4. (5 points) Use the Laplace transform to solve the initial-value problem

$$3y' + 5y = -2e^{-t}, \qquad y(0) = -2$$

5. (5 points) Find the inverse Laplace transform of the function $F(s) = \dfrac{se^{-s}}{s^2+9}$.

Create the piecewise definition of your solution that does not use the Heaviside function.

6. (5 points) Find the unit impulse response to the initial-value problem

$$y'' - 6y' + 9y = \delta(t), \qquad y(0) = y'(0) = 0$$

7. (5 points) For the initial-value problem $y' = ty,\ y(0) = 5$ calculate the first two iterations y_1 and y_2 of Euler's method if the step size is $h = 0.1$.

8. (10 points) Let $\bar{\mathsf{y}}(t) = C_1 e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ be a general solution of the system
$\bar{\mathsf{y}}' = A\bar{\mathsf{y}}$
for some matrix A.

Scketch a phase plane portrait of the system near the origin. Use arrows to indicate the direction of motion on all solution curves. Name the type of the equilibrium point (a saddle, a nodal sink, a nodal source, a center, a spiral sink, or a spiral source).

9. (10 points) By using the variation of parameters technique and the fundamental matrix find a particular solution to the system $\bar{\mathsf{y}}' = A\bar{\mathsf{y}} + \bar{\mathsf{f}}$,

where $A = \begin{pmatrix} 2 & 3 \\ 0 & -1 \end{pmatrix}$ and $\bar{\mathsf{f}} = \begin{pmatrix} 0 \\ 12 \end{pmatrix}$. The eigenvalues and associated eigenvectors are

$\lambda_1 = -1$, $\bar{\mathsf{v}}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\lambda_2 = 2$, $\bar{\mathsf{v}}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Write the answer as a single vector.

10. For the system of differential equations

$x' = x^2 - y^2$

$y' = (y-2)(y-x^2)$

(a) (3 points) find x-nullclines and y-nullclines.

(b) (3 points) calculate the coordinates of the equilibrium points.

(c) (3 points) Plot x-nullclines and y-nullclines. Use solid lines for x-nullclines and dashed lines
for y-nullclines. Plot the equilibrium points and label them with their coordinates.

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(d) (3 points) Use Jacobian to classify the equilibrium point $(-2, 2)$ (a saddle, a nodal source, etc.)

(e) (3 points) Use Jacobian to determine if the equilibrium point $(-2, 2)$ is stable or unstable.

11. (10 points) Expand the function $f(x) = x$ in a Fourier sine series valid on the interval $0 \leq x \leq 2\pi$.

12. (10 points) Find the temperature $u(t, x)$ in a rod modeled by the initial/boundary value problem

$$\begin{aligned} &u_t = u_{xx}, \quad \text{for } \ t > 0, \ \ 0 < x < 2\pi, \\ &u(0, t) = 0, \quad u(2\pi, t) = 0, \quad \text{for } \ t > 0, \\ &u(x, 0) = \frac{x}{4}, \quad \text{for } \ 0 \le x \le 2\pi. \end{aligned}$$

You may use results obtained in the previous problem.

Table of Laplace Transforms

$f(t)$	$L[f(t)] = F(s)$
1	$\frac{1}{s}$
$e^{at}f(t)$	$F(s-a)$
$H(t-a)$	$\frac{e^{-as}}{s}$
$f(t-a)H(t-a)$	$e^{-as}F(s)$
$\delta(t)$	1
$\delta(t-t_0)$	e^{-st_0}
$t^n f(t)$	$(-1)^n \frac{d^n F(s)}{ds^n}$
$f'(t)$	$sF(s) - f(0)$
$f * g$	$F(s)G(s)$
t^n	$\frac{n!}{s^{n+1}}$
$\sin bt$	$\frac{b}{s^2+b^2}$
$\cos bt$	$\frac{s}{s^2+b^2}$
e^{at}	$\frac{1}{s-a}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$