1. Solve the initial-value problem. Show all the work. Mention a type of the given differential equation.

   (a) $\dfrac{y'}{3} = x^2 y, \qquad y(0) = -8, \quad$ where $y' = \dfrac{dy}{dx}$.

   Solution: $\quad y(x) = -8e^{x^3}$

   (b) $t\dfrac{dx}{dt} = 4x + t^4, \quad x(1) = 5.$

   Solution: Integrating factor is $I = t^{-4}$, $x(t) = t^4 \ln t + Ct^4$, $C = 5$, $x(t) = (\ln t + 5)t^4$

   (c) $y'' - 4y = 2e^{4t}, \qquad y(0) = -3, \quad y'(0) = 11.$

   Solution: $\quad y(t) = c_1 e^{2t} + c_2 e^{-2t} \frac{1}{6} e^{4t}$

2. Find the general solution to the equation

$$y'' + 4y' + 4y = 12t^2 - 10.$$

Solution: $\quad y = C_1 e^{-2t} + C_2 t e^{-2t} + 3t^2 - 6t + 2$

3. A 1-kg mass when attached to a spring, stretches the spring to a distance of 4.9 m.

   (a) Calculate the spring constant.

   Solution: $\quad k = 2$

   (b) The system is plased in a viscous medium that supplies a damping constant $\mu = 3$ kg/s. The system is allowed to come to rest. Then the mass is displaced 1 m in the downward direction and given a sharp tap, imparting an instanteneous velocity of 1 m/s in the downward direction. Find the position of the mass as a function of time.

   Solution: see pb 16 in 4.4 (page 163) $\quad x'' + 3x' + 2x = 0, \quad x(0) = 1, \quad x'(0) = 1,$ $x(t) = 3e^{-t} - 2e^{-2t}$

4. Use Laplace transform to solve the IVP

$$y'' - y = e^t \cos t, \qquad y(0) = y'(0) = 0.$$

Solution: $\quad (s^2 - 1)Y = (s-1)(s+1)Y = \dfrac{s-1}{s^2 - 2s + 1}$

$$Y=\frac{1}{(s^2-2s+2)(s+1)}=-\frac{1}{5}\cdot\frac{s-3}{s^2-2s+2}+\frac{1}{5}\cdot\frac{1}{s+1}$$

$$=-\frac{1}{5}\cdot\frac{s-1}{(s-1)^2+1}+\frac{2}{5}\cdot\frac{1}{(s-1)^2+1}+\frac{1}{5}\cdot\frac{1}{s+1}$$

$y(t)=-\frac{1}{5}e^t\cos t+\frac{2}{5}e^t\sin t+\frac{1}{5}e^{-t}$

5. Find the Laplace transform of the function

$$g(t)=\begin{cases}3t & \text{for } 0\le t<2\\ 4 & \text{for } t\ge 2\end{cases}$$

Solution: $g(t)=3tH(t)-3(t-2)H(t-2)-2H(t-2),\quad L[g](s)=\frac{3}{s^2}-\frac{3}{s^2}e^{-2s}-\frac{2}{s}e^{-2s}$

6. Find the unit impulse response to the initial-value problem

$$y''-2y'+5y=\delta(t),\qquad y(0)=y'(0)=0$$

Solution: $E(s)=\frac{1}{s^2-2s+5}=\frac{1}{2}\cdot\frac{2}{(s-1)^2+2^2},\quad e(t)=\frac{1}{2}e^{-t}\sin 2t$

7. For the initial-value problem $y'=y+4t,\ y(0)=1$ calculate the first two iterations of Euler's method with step size $h=0.1$.

Solution: $t_0=0,\ y_0=1,\quad y_1=y_0+f(t_0,y_0)h=1+(1+4\cdot 0)(0.1)=1.1,$

$t_1=t_0+h=0.1,\quad y_2=y_1+f(t_1,y_1)h=1.1+(1.1+4\cdot 0.1)(0.1)=1.1+1.5\cdot 0.1=1.1+0.15=1.25$

8. Write the second-order equation as a system of two first-order equations

$$y''-e^{-2t}+3t^2y=\cos ty'.$$

Solution: $u_1=y,\ u_2=y'.$ The system is $u_1'=u_2,\ u_2'=e^{-2t}-3t^2u_1+(\cos t)u_2$

9. Find the general solution to the system. Write the answer in a vector form.

$$\begin{aligned}y_1'&=-3y_1-6y_2\\ y_2'&=-y_2\end{aligned}$$

Solution: $\lambda_1=-1,\ \bar{\mathrm{v}}_1=\begin{pmatrix}3\\-1\end{pmatrix},\quad \lambda_2=-3,\ \bar{\mathrm{v}}_2=\begin{pmatrix}1\\0\end{pmatrix}$

$\bar{\mathrm{y}}(t)=c_1e^{-t}\begin{pmatrix}3\\-1\end{pmatrix}+c_2e^{-3t}\begin{pmatrix}1\\0\end{pmatrix}$

10. By using the variation of parameters technique and the fundamental matrix find a particular solution to the system

$$y_1' = -3y_1 - 6y_2 + 2e^{-5t}$$
$$y_2' = -y_2$$

You may use results from the previous problem.

Solution: Skip this problem.

11. For the nonlinear system

$$x' = x(4y - 5)$$
$$y' = y(3 - x)$$

find all equilibrium points, classify their types and determine stability (stable, unstable or asymptotically stable).

Solution: EPs are $(0, 0)$ and $(3, 1.25)$.

$$J = \begin{pmatrix} 4y - 5 & 4x \\ -y & 3 - x \end{pmatrix}$$

At $(0, 0)$: $\lambda_1 = -5$, $\lambda_2 = 5$, saddle, unstable.

At $(3, 1.25)$: $\lambda = \pm i\sqrt{15}$, center, stability is inconclusive.

12. Expand the given function in a Fourier cosine series valid on the interval $0 \le x \le \pi$. Calculate a_0 separately.

$$f(x) = x.$$

Solution: $a_0 = \dfrac{2}{\pi}\displaystyle\int_0^\pi x\,dx = \pi$, $\quad a_n = \dfrac{2}{\pi}\displaystyle\int_0^\pi x\cos(nx)\,dx = \dfrac{2((-1)^n - 1)}{\pi n^2}$

If $n = 2k$ (even) then $a_{2k} = 0$.

If $n = 2k + 1$ (odd) then $a_{2k+1} = \dfrac{2(-2)}{\pi(2k+1)^2} = -\dfrac{4}{\pi}\dfrac{1}{(2k+1)^2}$

$$x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) = \frac{a_0}{2} + \sum_{k=0}^{\infty} a_{2k+1}\cos((2k+1)x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos((2k+1)x)$$

(The solution $x = \dfrac{\pi}{2} + \dfrac{2}{\pi}\displaystyle\sum_{k=1}^{\infty}\dfrac{(-1)^n - 1}{n^2}\cos(nx)$ is also correct.)

13. Find the temperature $u(t, x)$ in a rod modeled by the initial/boundary value problem

$$\begin{aligned} &u_t = 0.03\, u_{xx}, \quad \text{for } \ t > 0, \ \ 0 < x < \pi, \\ &u_x(0, t) = u_x(\pi, t) = 0, \quad \text{for } \ t > 0, \\ &u(x, 0) = x, \quad \text{for } \ 0 \le x \le \pi. \end{aligned}$$

You may use results obtained in the previous problem.

Solution: It is a Neumann's problem with $k = 0.03$ and $L = \pi$. Then $\omega_n = n$ and $\lambda_n = n^2$. Its solution is

$$u(x, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\, e^{-0.03 n^2 t} \cos(nx)$$

where a_n are coefficients of the Fourier Cosine series of the function $f(x) = x$ obtained in the previous problem. Therefore the solution is

$$u(x, t) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}\, e^{-0.03(2k+1)^2 t} \cos((2k+1)x)$$

(The solution $u(x, t) = \dfrac{\pi}{2} + \dfrac{2}{\pi} \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^n - 1}{n^2}\, e^{-0.03 n^2 t} \cos(nx)$ is also correct.)