MATH 0230 - Analytic Geometry and Calculus II
SAMPLE FINAL EXAM

Time: 1 hour 50 min

1. Evaluate the integral if it converges. If it diverges, show divergence.

(a) $I = \int_{-\infty}^{0} xe^{-3x^2}\,dx = \lim_{t\to\infty} \int_{-t}^{0} xe^{-3x^2}dx$

$u = -3x^2, \quad du = -6x\,dx, \quad u(-t) = -3t^2, \quad u(0) = 0$

$I = \lim_{t\to\infty} \int_{-3t^2}^{0} \left(-\frac{1}{6}\right) e^u du = -\frac{1}{6} \lim_{t\to\infty} \left(1 - e^{-3t^2}\right)$

$= -\frac{1}{6}(1-0) = -\frac{1}{6}$

(b) $I = \int_{3}^{7} \frac{5}{(3-x)^3}\,dx$ discontinuity at $x = 3$

$I = \lim_{t\to 3^+} \int_{t}^{7} \frac{5}{(3-x)^3}dx$ $\quad u = 3-x, \quad du = -dx$
$u(t) = 3-t, \quad u(7) = -4$

$I = \lim_{t\to 3^+} \int_{3-t}^{-4} (-5)u^{-3}du = \lim_{t\to 3^+} \frac{5}{2} u^{-2} \Big|_{3-t}^{-4}$

$= \frac{5}{2} \lim_{t\to 3^+} \left[\frac{1}{16} - \frac{1}{(3-t)^2}\right] = -\infty$

diverges

2. At what points on the curve $x = 2t^3$, $y = 3t^2 + 9t - 5$ does the tangent line have slope $\frac{1}{2}$?

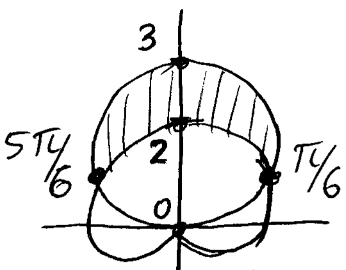
$$m = \frac{dy/dt}{dx/dt} = \frac{6t+9}{6t^2} = \frac{2t+3}{2t^2} = \frac{1}{2}, \quad 2t+3 = t^2$$

$$t^2 - 2t + 3 = 0, \quad (t+1)(t-3) = 0, \quad t = -1, \ t = 3$$

$$t = -1 \Rightarrow (x, y) = (-2, -11)$$

$$t = 3 \Rightarrow (x, y) = (54, 49)$$

3. Find the area of the region that lies inside the curve $r = 3\sin\theta$ and outside the curve $r = 1 + \sin\theta$.



$$3\sin\theta = 1 + \sin\theta, \quad \sin\theta = \frac{1}{2}$$

$$\theta_1 = \frac{\pi}{6}, \quad \theta_2 = \frac{5\pi}{6}$$

$$A = \frac{1}{2}\int_{\frac{\pi}{6}}^{5\pi/6} \left((3\sin\theta)^2 - (1+\sin\theta)^2 \right) d\theta$$

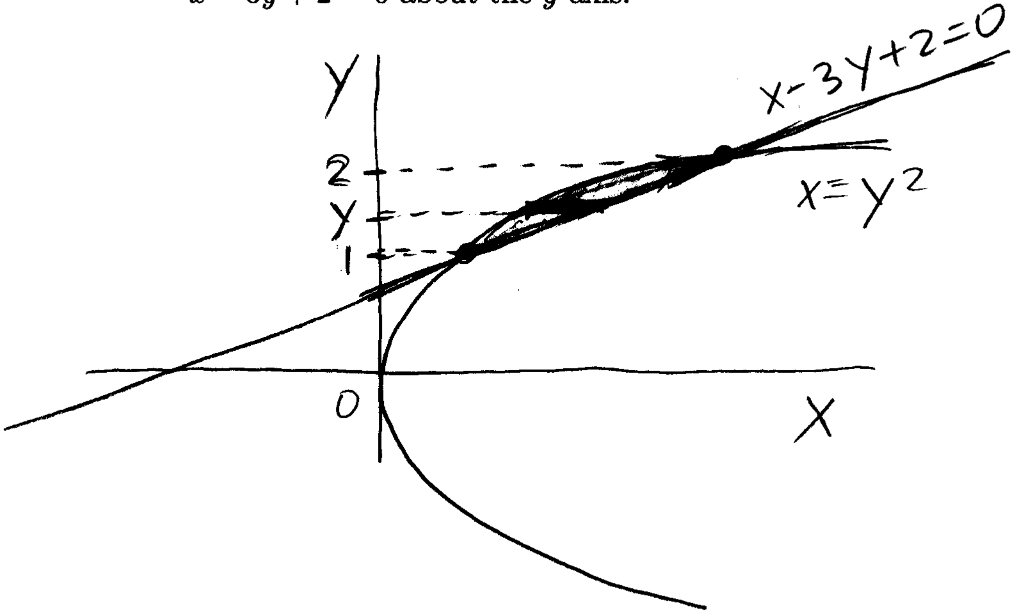
$$A = \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left(8\sin^2\theta - 2\sin\theta - 1 \right) d\theta = \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left(4(1-\cos 2\theta) - 2\sin\theta - 1 \right) d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left(-4\cos 2\theta - 2\sin\theta + 3 \right) d\theta = \frac{1}{2}\Big[-2\sin 2\theta + 2\cos\theta + 3\theta \Big]_{\frac{\pi}{6}}^{\frac{5\pi}{6}}$$

$$= \frac{1}{2}\left[-2\sin\frac{5\pi}{3} + 2\cos\frac{5\pi}{6} + \frac{5\pi}{2} + 2\sin\frac{\pi}{3} - 2\cos\frac{\pi}{6} - \frac{\pi}{2} \right]$$

$$= \frac{1}{2}\left[\sqrt{3} - \sqrt{3} + \frac{4\pi}{2} + \sqrt{3} - \sqrt{3} \right] = \pi$$

4. Find the volume generated by rotating the region bounded by the curves $x = y^2$ and $x - 3y + 2 = 0$ about the y-axis.



$$x = 3y+2 = y^2, \quad y^2 - 3y - 2 = 0, \quad (y-1)(y-2) = 0$$

$$y = 1, \; y = 2$$

Method of washers:

$$\Delta V = \pi\left[(3y-2)^2 - (y^2)^2\right]\Delta y$$

$$V = \pi \int_1^2 \left[9y^2 - 12y + 4 - y^4\right] dy$$

$$= \pi\left[3y^3 - 6y^2 + 4y - \frac{1}{5}y^5\right]_1^2$$

$$V = \pi\left[24 - 24 + 8 - \frac{32}{5} - 3 + 6 - 4 + \frac{1}{5}\right]$$

$$= \pi\left[7 - \frac{31}{5}\right] = \frac{4}{5}\pi$$

5. Find the volume of the parallelepiped with adjacent edges PQ, PR, and PS if $P(3,0,1)$, $Q(-1,2,5)$, $R(5,1,-1)$, and $S(0,4,2)$.

$$\vec{PQ} = \langle -4, 2, 4 \rangle, \quad \vec{PR} = \langle 2, 1, -2 \rangle, \quad \vec{PS} = \langle -3, 4, 1 \rangle$$

$$\vec{PR} \times \vec{PS} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & 1 & -2 \\ -3 & 4 & 1 \end{vmatrix} = \langle 9, 4, 11 \rangle$$

$$V = |\vec{PQ} \cdot (\vec{PR} \times \vec{PS})| = |-36 + 8 + 44| = 16$$

6. Find parametric equations for the line through $(2,4,6)$ that is perpendicular to the plane $x - y + 3z = 7$. In what points does the line intersect the coordinate planes?

$$\bar{v} = \bar{n} = \langle 1, -1, 3 \rangle$$

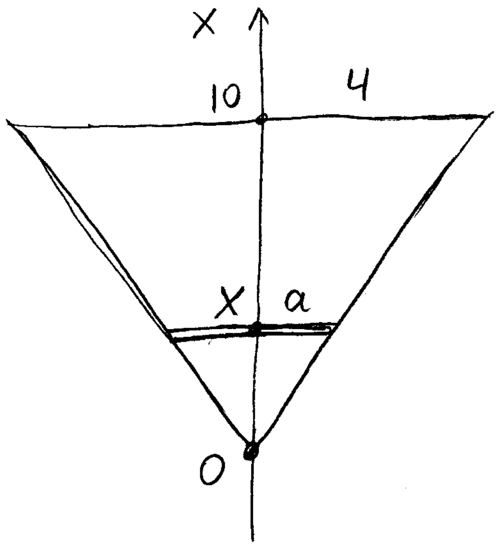
$$x = 2 + t, \quad y = 4 - t, \quad z = 6 + 3t$$

xy-plane, $z = 0$: $t = -2$, $(0, 6, 0)$

xz-plane, $y = 0$: $t = 4$, $(6, 0, 18)$

yz-plane, $z = 0$: $t = -2$, $(0, 6, 0)$

7. A water tank is in the shape of an inverted cone (with its sharp end down). The height of the cone is 10 meters and the diameter of the base is 8 meters. The tank is full of water. Find the work required to pump all of the water out over the side. You may use $\rho = 1000\frac{\text{kg}}{\text{m}^3}$ for the density of water and $g = 9.8\frac{\text{m}}{\text{s}^2}$ for the acceleration due to gravity.



$$W = Fd = mgd = \rho V g d$$

$$W = \rho g \, V d$$

$$\Delta V = \pi a^2 \Delta X$$

$$\frac{a}{4} = \frac{X}{10} \quad , \quad a = \frac{4}{10} X = \frac{2}{5} X$$

$$\Delta V = \pi \cdot \frac{4}{25} X^2 \Delta X \quad , \quad d = 10 - X$$

$$\Delta W = \rho g \cdot \frac{4}{25} \pi X^2 (10 - X) \Delta X$$

$$W = \int_0^{10} \frac{4 \rho g}{25} \pi \left(10 X^2 - X^3\right) dX$$

$$= \frac{4 \cdot 9.8 \cdot 10^3}{25} \pi \left[\frac{10}{3} X^3 - \frac{X^4}{4}\right]_0^{10} = 16 \cdot 9.8 \cdot 10 \pi \left[\frac{10^4}{3} - \frac{10^4}{4}\right]$$

$$= 16 \cdot 9.8 \cdot 10^5 \pi \cdot \frac{1}{12} = \frac{4}{3} \cdot 9.8 \cdot 10^5 \pi \quad J$$

8. Determine whether the series is convergent or divergent. Clearly show reason by a valid test.

(a) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{\sqrt{n^2+3}}$ $\qquad a_n=(-1)^{n-1}\frac{n}{\sqrt{n^2+3}}$

if n is odd then $a_n=-\frac{n}{\sqrt{n^2+3}}$ and $\lim_{n\to\infty} a_n=-1$

if n is even then $a_n=\frac{n}{\sqrt{n^2+3}}$ and $\lim_{n\to\infty} a_n=1$

Hence, $\lim_{n\to\infty} a_n$ DNE $\Rightarrow$ the series is divergent by the divergence test.

(b) $\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n(n+3)$

$a_n=\left(\frac{3}{4}\right)^n(n+3)=\frac{3^n(n+3)}{4^n}$

Ratio Test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{3^{n+1}(n+4)}{4^{n+1}}\cdot\frac{4^n}{3^n(n+3)}$

$=\lim_{n\to\infty}\frac{3}{4}\cdot\frac{n+4}{n+3}=\frac{3}{4}<1\Rightarrow$ The series is convergent.

9. For the series

$$\sum_{n=1}^{\infty}(-1)^n\frac{2n}{n^2+1} \qquad a_n=(-1)^n\frac{2n}{n^2+1}$$

determine whether it is absolutely convergent, conditionally convergent or divergent. Clearly show reason by a valid test.

Alternating series. $b_n=\frac{2n}{n^2+1}>0$, (1) Let $f(x)=\frac{2x}{x^2+1}$,

$f'(x)=\frac{2(1-x^2)}{(x^2+1)^2}<0$ for $x>1 \Rightarrow f(x)$ is decreasing $\Rightarrow b_n>b_{n+1}$

(2) $\lim_{n\to\infty} b_n=0$. By Alternat. Series Test the series is convergent

$|a_n|=\frac{2n}{n^2+1}$. Consider $\sum_{n=1}^{\infty}\frac{1}{n}$ with $c_n=\frac{1}{n}$ (harmonic series)

Lim Comparison: $\lim_{n\to\infty}\left|\frac{a_n}{c_n}\right|=\lim_{n\to\infty}\frac{2n^2}{n^2+1}=2\Rightarrow$ The given series is not abs. convergent. It is condit conv.

10. Determine the sum of the given series.

(a) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^n}{n}$ $= \ln(1+2) = \ln 3$

(see series on page 484)

(b) $\sum_{n=1}^{\infty}\frac{2}{n^2+2n}$ $\quad a_n = \frac{2}{n(n+2)} = \frac{1}{n} - \frac{1}{n+2}$. telescoping

partial m^{th} sum: $S_m = \left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\cdots$

$+\left(\frac{1}{m-2}-\frac{1}{m}\right)+\left(\frac{1}{m-1}-\frac{1}{m+1}\right)+\left(\frac{1}{m}-\frac{1}{m+2}\right) = 1+\frac{1}{2}-\frac{1}{m+1}-\frac{1}{m+2}$

$\lim_{m\to\infty} S_m = 1+\frac{1}{2} = \frac{3}{2}$

11. Determine the interval of convergence of the series

$$\sum_{n=1}^{\infty}\frac{(x-2)^n}{3^n\sqrt{n}} \qquad c_n = \frac{1}{3^n\sqrt{n}}$$

$R = \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty}\frac{3^{n+1}\sqrt{n+1}}{3^n\sqrt{n}} = 3$. $|x-2|<3 \Rightarrow x\in(-1,5)$

End points: -1: $a_n = \frac{(-3)^n}{3^n\sqrt{n}} = \frac{(-1)^n}{\sqrt{n}}$. Alternat. series with $b_n = \frac{1}{\sqrt{n}} > 0$

$b_{n+1} < b_n$, $\lim_{n\to\infty} b_n = 0$. The series is convergent.

5: $a_n = \frac{3^n}{3^n\sqrt{n}} = \frac{1}{n^{1/2}}$ p-series with $p=\frac{1}{2}$, diverges

Interval of converges is $[-1,5)$

12. For the functions below, determine their Maclaurin series. You may apply any general series you know. You should use sigma $\sum$ notation.

(a) $\frac{3}{(1-3x)^2} = \frac{d}{dx}\left(-\frac{1}{1-3x}\right) = \frac{d}{dx}\left[-\sum_{n=0}^{\infty}(3x)^n\right]$

$= -\frac{d}{dx}\left[\sum_{n=0}^{\infty} 3^n x^n\right] = -\sum_{n=1}^{\infty} n 3^n x^{n-1}$

(b) $x e^{-3x} = x\sum_{n=0}^{\infty}\frac{(-3x)^n}{n!} = \sum_{n=0}^{\infty}(-1)^n \frac{3^n}{n!} x^{n+1}$

$= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{3^{n-1}}{(n-1)!}x^n$

13. For the functions below, determine their Taylor series about given values of a. You may apply any general series you know. You should use sigma $\sum$ notation.

(a) $f(x) = x^4 - 5x^2 + 8x - 4, \quad a = 1$ $\quad f(1)=0,\ f'=4x^3-10x+8,\ f'(1)=2$

$f''=12x^2-10,\ f''(1)=2,\ f'''=24x,\ f'''(1)=24,\ f^{IV}=24,\ f^{IV}(1)=24$

$f(x) = 0 + 2(x-1) + \frac{2}{2!}(x-1)^2 + \frac{24}{3!}(x-1)^3 + \frac{24}{4!}(x-1)^4$

$f(x) = 2(x-1) + (x-1)^2 + 4(x-1)^3 + (x-1)^4$

(b) $\cos 2x, \quad a = \frac{\pi}{4}$

$f(\frac{\pi}{4})=0,\ f'=-2\sin 2x,\ f'(\frac{\pi}{4})=-2,\ f''=-4\cos 2x,\ f''(\frac{\pi}{4})=0,\ f'''=8\sin 2x$

$f'''(\frac{\pi}{4})=8,\ f^{IV}=16\cos 2x = 16f,\ f^{IV}(\frac{\pi}{4})=0,\quad f^{(n+4)}=16f^{(n)}$

$\cos 2x = 0 - \frac{2}{1!}(x-\frac{\pi}{4}) - 0 + \frac{8}{3!}(x-\frac{\pi}{4})^3 + 0 - \frac{16\cdot 2}{5!}(x-\frac{\pi}{4})^5 + \cdots$

$\cos 2x = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{2^{2n+1}}{(2n+1)!}\left(x-\frac{\pi}{4}\right)^{2n+1}$ (even terms all are 0)

14. Solve the initial-value problem. Show all the work. Mention a type of the given differential equation.

(a) $y' = \dfrac{\sin x}{2y}, \qquad y(\pi) = 2.$

$2yy' = \sin x, \quad \int 2y\,dy = \int \sin x\,dx, \quad y^2 = -\cos x + C$

$y(\pi) = 2 > 0 \Rightarrow \quad y = \sqrt{-\cos x + C} \quad (\text{not } -\sqrt{-\cos x + C})$

$y(\pi) = \sqrt{1 + C} = 2 \Rightarrow C = 3$

$y(x) = \sqrt{-\cos x + 3}$

(b) $y' + \dfrac{3}{x} y = \dfrac{2}{x^2}, \qquad y(1) = 5.$ Linear 1^{st} Order Equation

Integr. Factor: $I = e^{\int \frac{3}{x} dx} = e^{3 \ln x} = e^{\ln x^3} = x^3$

$x^3 y' + 3x^2 y = 2x, \quad (x^3 y)' = 2x, \quad x^3 y = x^2 + C$

$y = x^{-1} + Cx^{-3}, \quad y(1) = 1 + C = 5 \Rightarrow C = 4$

$y(x) = x^{-1} + 4x^{-3}$

(c) $y'' - 2y' + 10y = 0, \qquad y(0) = 0, \quad y'(0) = 6.$

Char. eq. $r^2 - 2r + 10 = 0 \quad D = 4 - 40 = -36 < 0$

$\alpha = \frac{2}{2} = 1, \quad \beta = \frac{\sqrt{-D}}{2} = 3$

$y(x) = e^x (C_1 \cos 3x + C_2 \sin 3x), \quad y(0) = C_1 = 0$

$y(x) = C_2 e^x \sin 3x, \quad y'(x) = C_2 e^x \sin 3x + 3C_2 e^x \cos 3x$

$y'(0) = 3C_2 = 6 \Rightarrow C_2 = 2 \qquad y(x) = 2e^x \sin 3x$

15. Find the general solution to the second-order nonhomogeneous differential equation

$$y'' + 4y = 8\cos 2x$$

Use the method of undetermined coefficients.

Char. eq: $r^2 + 4 = 0$, $D = -16 < 0$

$\alpha = 0$, $\beta = \frac{\sqrt{-D}}{2} = \frac{4}{2} = 2$

$y_c(x) = C_1 \cos 2x + C_2 \sin 2x$

$y_p(x) = x(a\cos 2x + b \sin 2x)$

Let $g = a\cos 2x + b\sin 2x$. Then $g' = -2a\sin 2x + 2b\cos 2x$

$g'' = -4g$

$y_p(x) = xg$

$y_p'(x) = (xg)' = g + xg'$

$y_p''(x) = (g + xg')' = g' + g' + xg'' = 2g' - 4xg$

$= 2g' - 4y_p$

Then $y_p'' + 4y_p = 2g' - 4y_p + 4y_p = 2g'$

So, $2g' = 8\cos 2x$, $g' = 4\cos 2x$

$-2a\sin 2x + 2b\cos 2x = 4\cos 2x \Rightarrow a = 0$, $b = 2$

$y_p(x) = 2x \sin 2x$

General solution: $y(x) = y_c(x) + y_p(x)$

$y(x) = C_1\cos 2x + C_2 \sin 2x + 2x\sin 2x$