MATH 0230 - Analytic Geometry and Calculus II
Solutions to SAMPLE FINAL EXAM 1

1. (10 points) $I = \int_0^{\pi/2} \sin^2 x \cos^2 x \, \sin x \, dx = \int_0^{\pi/2} (1 - \cos^2 x) \cos^2 x \, \sin x \, dx$ **+3 pts**

Substitution: $u = \cos x, \;\; du = -\sin x \, dx, \;\; u(0) = 1, \;\; u(\pi/2) = 0.$ **+3 pts**

Then $I = \int_1^0 (1 - u^2) u^2 \, (-du) = \int_0^1 (u^2 - u^4) \, du$ **+2 pts**

$= \left[\frac{u^3}{3} - \frac{u^5}{5} \right|_0^1 = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}$ **+2 pts**

2. (a) (10 points) There is a discontinuity of the integrand at $x = 7$. **+2 pts**

Hence, $I = \lim_{t \to 7^-} \int_3^t \frac{1}{\sqrt{7-x}}\, dx$ **+2 pts**

Substitution: $u = 7 - x,\ du = -dx,\ u(3) = 4,\ u(t) = 7 - t.$ **+2 pts**

Then $I = \lim_{t \to 7^-} \int_4^{7-t} u^{-1/2}\,(-du)$ **+2 pts**

$= -2 \lim_{t \to 7^-} \left[u^{1/2}\right]_4^{7-t} = -2 \lim_{t \to 7^-} \left(\sqrt{7-t} - 2\right) = -2\,(0-2) = 4$ **+2 pts**.

(b) (10 points) $I = \lim_{t \to \infty} \int_1^t \frac{x}{4+x^2}\, dx$ **+2 pts**

Substitution: $u = 4 + x^2,\ du = 2x dx,\ u(1) = 5,\ u(t) = 4 + t^2.$ **+2 pts**

Then $I = \lim_{t \to \infty} \int_5^{4+t^2} \frac{1}{2} \cdot \frac{du}{u}$ **+1 pts**

$= \frac{1}{2} \lim_{t \to \infty} \ln|u| \Big|_5^{4+t^2} = \frac{1}{2} \lim_{t \to \infty} (\ln(4+t^2) - \ln 5)$ **+2 pts**

$= \infty.$ **+1 pts**

So, the integral diverges. **+2 pts**

3. (10 points) The curves intersect at the points $(0,0)$ and $(2,4)$, so that $0 \leq x \leq 2$. **+2 pts**

We use vertical rectangles to obtain cylindrical shells.

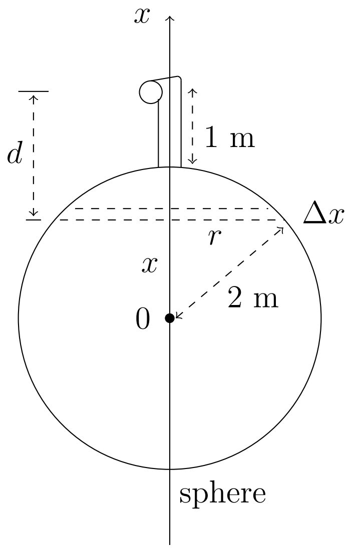
Then $r = x$, $\Delta r = \Delta x$, $h = 4x - x^2$. **+2 pts**

$\Delta V = 2\pi x\,(4x - x^2)\,\Delta x = 2\pi(4x^2 - x^3)\Delta x$ **+2 pts**

$$V = 2\pi \int_0^2 (4x^2 - x^3)\,dx = 2\pi \left[\frac{4x^3}{3} - \frac{x^4}{4}\right]_0^2$$ **+2 pts**

$$= 2\pi \left[\frac{32}{3} - 4\right] = \frac{40\pi}{3}$$ **+2 pts**

4. (10 points)



$r^2 + x^2 = 2^2, \quad r^2 = 4 - x^2$ **+2 pts**

$\Delta V = \pi r^2 \Delta x = \pi(4 - x^2)\Delta x$ **+2 pts**

$\Delta F = \rho g \Delta V = 9800\pi(4 - x^2)\Delta x$ **+1 pts**

$d = 2 + 1 - x = 3 - x$ **+1 pts**

$\Delta W = \Delta F \cdot d = 9800\pi(4 - x^2)(3 - x)\Delta x$ **+1 pts**

$-2 \le x \le 2$ **+1 pts**

$W = \int_{-2}^{2} 9800\pi(4 - x^2)(3 - x)\, dx$ **+2 pts**

5. (a) (5 points) $\bar{\mathsf{a}} = \bar{\mathsf{PQ}} = \langle 1, 1, -2 \rangle$, $\bar{\mathsf{b}} = \bar{\mathsf{PR}} = \langle -3, 0, 3 \rangle$ **+1 pts**

$\bar{\mathsf{a}} \cdot \bar{\mathsf{b}} = -9$ **+1 pts**

$|\bar{\mathsf{a}}| = \sqrt{6}, \quad |\bar{\mathsf{b}}| = \sqrt{18}$ **+1 pts**

$$\cos\theta = \frac{\bar{\mathsf{a}} \cdot \bar{\mathsf{b}}}{|\bar{\mathsf{a}}|\,|\bar{\mathsf{b}}|} = -\frac{9}{\sqrt{6}\sqrt{18}}$$

$$\left(\text{Possible answers:} \quad \cos\theta = -\frac{9}{\sqrt{108}} = -\frac{\sqrt{3}}{2}\right)$$ **+1 pts**

$$\theta = \cos^{-1}\left(-\frac{9}{\sqrt{108}}\right) \quad \left(\text{or } \theta = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right).$$ **+1 pts**

(b) (5 points)

$$\bar{\mathsf{a}} \times \bar{\mathsf{b}} = \begin{vmatrix} \bar{\mathsf{i}} & \bar{\mathsf{j}} & \bar{\mathsf{k}} \\ 1 & 1 & -2 \\ -3 & 0 & 3 \end{vmatrix} = \langle 3, 3, 3 \rangle$$

+3 pts

Area is $A = \frac{1}{2}|\bar{\mathsf{a}} \times \bar{\mathsf{b}}| = \frac{\sqrt{3 \cdot 3^2}}{2} = \frac{3\sqrt{3}}{2}$. **+2 pts**

(c) (5 points) The vectors $\bar{\mathsf{a}} = \bar{\mathsf{PQ}} = \langle 1, 1, -3 \rangle$ and $\bar{\mathsf{b}} = \bar{\mathsf{PR}} = \langle -5, 0, 3 \rangle$ **+1 pts**

are on the plane.

Their cross product is orthogonal to the plane. The normal vector is

$\bar{\mathsf{n}} = \frac{1}{3}\bar{\mathsf{a}} \times \bar{\mathsf{b}} = \langle 1, 1, 1 \rangle$ **+2 pts**

With the point P and normal vector $\bar{\mathsf{n}}$, an equation of the plane is

$x - 1 + y + 3 + z - 1 = 0$ or $x + y + z + 1 = 0$. **+2 pts**

6. (10 points) $x(0) = 0,\ y(0) = 1$ **+2 pts**

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2(t+1)}{\cos t}$$ **+4 pts**

$\dfrac{dy}{dx} = 2$ when $t = 0.$ **+2 pts**

The equation of the tangent line is $y = 2x + 1.$ **+2 pts**

7. (a) (10 points) $2\sin 5\theta = 0,\quad \theta = 0,\ \theta = \frac{\pi}{5}.$ Hence $0 \le \theta \le \frac{\pi}{5}.$ **+2 pts**

$$A = \frac{1}{2}\int_0^{\frac{\pi}{5}} 4\sin^2 5\theta\, d\theta = \int_0^{\frac{\pi}{5}} 2\sin^2 5\theta\, d\theta$$ **+2 pts**

$$= \int_0^{\frac{\pi}{5}} (1 - \cos 10\theta)\, d\theta$$ **+2 pts**

$$= \left[\theta - \frac{\sin 10\theta}{10}\right]_0^{\frac{\pi}{5}}$$ **+2 pts**

$$= \frac{\pi}{5}$$ **+2 pts**

(b) (5 points) $\dfrac{dr}{d\theta} = 10\cos 5\theta.$ **+2 pts**

$$L = \int_0^{\frac{\pi}{5}} \sqrt{4\sin^2 5\theta + 100\cos^2 5\theta}\, d\theta$$ **+3 pts**

8. (a) (5 points) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{5^n} = \sum_{n=1}^{\infty} \frac{2}{5} \cdot \left(\frac{2}{5}\right)^{n-1}$ **+1 pts**

It is a geometric series with $a = \frac{2}{5}$ and $r = \frac{2}{5}$. **+1 pts**

It is convergent because $|r| < 1$ **+1 pts**

Its sum is $\displaystyle\frac{a}{1-r} = \frac{2/5}{1-2/5} = \frac{2}{3}$. **+2 pts**

(b) (5 points) $a_n = \dfrac{2^n}{n^2}$ **+1 pts**

Ratio test: $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)^2} \cdot \frac{n^2}{2^n} = \lim_{n\to\infty} \frac{2n^2}{n^2+2n+1} = 2.$ **+2 pts**

The series is divergent because the limit is greater than 1. **+2 pts**

(c) (5 points) It is an alternating series with $b_n = \dfrac{1}{n+2} > 0$. **+1 pts**

(1) $b_{n+1} = \dfrac{1}{n+3} < \dfrac{1}{n+2} = b_n$, decreasing **+1 pts**

(2) $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n+2} = 0.$ **+1 pts**

So, the conditions (1) and (2) of the alternating series test hold and, hence,

the series is convergent. **+2 pts**

9. (a) (10 points) $\dfrac{1}{n^2+3n+2} = \dfrac{1}{(n+1)(n+2)} = \dfrac{1}{n+1} - \dfrac{1}{n+2}$ **+3 pts**

$S_n = \displaystyle\sum_{i=0}^{n} \frac{1}{i^2+3i+2} = \sum_{i=0}^{n}\left(\frac{1}{i+1} - \frac{1}{i+2}\right) = \left(1-\tfrac{1}{2}\right) + \left(\tfrac{1}{2}-\tfrac{1}{3}\right) + \left(\tfrac{1}{3}-\tfrac{1}{4}\right) + \left(\tfrac{1}{4}-\tfrac{1}{5}\right)$

$+\cdots+\left(\dfrac{1}{n} - \dfrac{1}{n+1}\right) + \left(\dfrac{1}{n+1} - \dfrac{1}{n+2}\right)$ **+3 pts**

$= 1 - \dfrac{1}{n+2}$ **+2 pts**

$\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2+3n+2} = \lim_{n\to\infty} S_n = 1.$ **+2 pts**

(b) (5 points) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{5^n\, n!} = \sum_{n=1}^{\infty} \frac{\left(\frac{2}{5}\right)^n}{n!}$ **+3 pts**

$= e^{\frac{2}{5}}.$ **+2 pts**

10. (10 points) $c_n = \dfrac{(-1)^n}{n\,4^n}$ **+1 pts**

$R = \displaystyle\lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty} \frac{(n+1)\,4^{n+1}}{n\,4^n} = 4.$ **+3 pts**

So, the series converges in the interval $(-4, 4)$. **+2 pts**

End points.

$x = -4$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\,(-4)^n}{n\,4^n} = \sum_{n=1}^{\infty} \frac{1}{n}$. It is the harmonic series and hence, diverges. **+1 pts**

$x = 4$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\,4^n}{n\,4^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$

It is the alternating harmonic series and hence, converges. **+1 pts**

The interval of convergence is $(-4, 4]$. **+2 pts**

11. (a) (5 points) $\dfrac{1}{5+x} = \dfrac{1}{5\left(1+\dfrac{x}{5}\right)}$ **+1 pts**

$= \dfrac{1}{5} \cdot \dfrac{1}{1-\left(-\dfrac{x}{5}\right)}$ **+2 pts**

$= \dfrac{1}{5} \sum_{n=0}^{\infty} \left(-\dfrac{x}{5}\right)^n$ **+1 pts**

$= \sum_{n=0}^{\infty} \dfrac{(-1)^n}{5^{n+1}} \, x^n$ **+1 pts**

(b) (5 points) $\ln(5+x) = \displaystyle\int \frac{1}{5+x}\, dx$ **+2 pts**

$= \displaystyle\int \sum_{n=0}^{\infty} \frac{(-1)^n}{5^{n+1}} \, x^n \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)\, 5^{n+1}} \, x^{n+1} + C$ **+1 pts**

Put $x = 0$. Then $\ln 5 = C$. **+1 pts**

Hence,

$\ln(5+x) = \ln 5 + \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)\, 5^{n+1}} \, x^{n+1} = \ln 5 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\, 5^n} \, x^n$ **+1 pts**

Alternative way: $\ln(5+x) = \ln\left(5\left(1+\dfrac{x}{5}\right)\right) = \ln 5 + \ln\left(1+\dfrac{x}{5}\right)$ **+3 pts**

$= \ln 5 + \sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n} \left(\dfrac{x}{5}\right)^n = \ln 5 + \sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n\, 5^n} \, x^n$ **+2 pts**

(c) (5 points) $-x^2 \cos x = -x^2 \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \, (2x)^{2n}$ **+3 pts**

$= \sum_{n=0}^{\infty} \dfrac{(-1)^{n+1}\, 4^n}{(2n)!} \, x^{2n+2}$ **+2 pts**

12. (10 points) $\sin x = \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi)}{n!} (x-\pi)^n$ **+2 pts**

$f(\pi) = 0, \quad c_0 = 0$ **+1 pts**

$f'(x) = \cos x, \quad f'(\pi) = -1, \quad c_1 = -\frac{1}{1!}$ **+1 pts**

$f''(x) = -\sin x, \quad f''(\pi) = 0, \quad c_2 = 0$ **+1 pts**

$f'''(x) = -\cos x, \quad f'''(\pi) = 1, \quad c_3 = -\frac{1}{3!}$ **+1 pts**

$f^{(4)}(x) = \sin x = f(x), \quad f^{(4)}(\pi) = 0, \quad c_4 = 0$ **+1 pts**

and etc.

Hence, $\sin x = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)!} (x-\pi)^{2n-1}$ **+3 pts**

13. (a) (10 points) It is a separable equation **+2 pts**

$\frac{dy}{dx} = 3x^2 y, \quad \frac{dy}{y} = 3x^2\,dx, \quad \int \frac{dy}{y} = \int 3x^2\,dx$ **+3 pts**

$\ln|y| = x^3 + C_1, \quad |y| = e^{C_1}\,e^{x^3}, \quad y = Ce^{x^3}, \quad C = \pm\,e^{C_1}.$ **+3 pts**

$y(0) = C = 5, \quad y = 5e^{x^3}\,.$ **+2 pts**

(b) (10 points) It is a linear differential equation. **+2 pts**

$y' - \frac{2}{x}\,y = x^2.$ **+1 pts**

The integrating factor is $I(x) = e^{\int -\frac{2}{x}\,dx} = e^{-2\ln|x|} = x^{-2}.$ **+2 pts**

Then

$x^{-2}y' - 2x^{-3}y = 1, \quad (x^{-2}y)' = 1, \quad x^{-2}y = \int dx = x + C, \quad y = x^3 + Cx^2.$ **+3 pts**

$y(1) = 1 + C = 4, \quad C = 3, \quad y(x) = x^3 + 3x^2\,.$ **+2 pts**

(c) (10 points) It is a second-order linear homogeneous differential equation. **+2 pts**

Its characteristic equation $r^2 + 6r + 9 = 0$ has one root $r = -3.$ **+2 pts**

The general solution is $y(x) = e^{-3x}\,(c_1 + c_2 x)$ **+3 pts**

Initial conditions:

$y(0) = c_1 = 2, \quad y'(x) = -3e^{-3x}(2 + c_2 x) + c_2 e^{-3x}$, $y'(0) = -6 + c_2 = -5, \quad c_2 = 1.$ **+2 pts**

$y(x) = e^{-3x}\,(2 + x)\,.$ **+1 pts**

14. (15 points) The characteristic equation of the complementary equation

$r^2 + 2r + 5 = 0$ has two roots $r_1 = -1 + 2i$ and $r_2 = -1 - 2i$. **+2 pts**

The solution of the complementary equation is $y_c(x) = e^{-x}\,(c_1 \cos 2x + c_2 \sin 2x)$. **+2 pts**

For a particular solution we try $y_p(x) = (Ax + B)\,e^x$. **+2 pts**

Then $y_p{}' = Ae^x + (Ax + B)\,e^x$, $y_p'' = 2Ae^x + (Ax + B)\,e^x$. **+2 pts**

Substituting into the differential equation, we get:

$4Ae^x + 8\,(Ax + B)\;e^x = -16xe^x$ **+2 pts**

$8A = -16, \quad 4A + 8B = 0, \quad A = -2, \quad B = 1.$ **+2 pts**

$y_p = (-2x + 1)\,e^x$ **+1 pts**

The general solution $y(x) = y_c(x) + y_p(x)$ is

$y(x) = e^{-x}\,(c_1 \cos 2x + c_2 \sin 2x) + (-2x + 1)\,e^x$ **+2 pts**