MATH 0120 - Business Calculus SAMPLE FINAL EXAM

Solutions

1. [5 points] Find an equation of the line that passes through the points $(-2,3)$ and $(4,6)$

Solution: $m=\dfrac{6-3}{4+2}=\dfrac{1}{2}$, $b=3-\dfrac{1}{2}\cdot(-2)=4$. Equation: $y=\dfrac{1}{2}x+4$.

2. Evaluate the limit

(a) [5 points] $\displaystyle\lim_{x\to 0}\frac{x^2-2x}{x^2+2x}$

Solution: $$\lim_{x\to 0}\frac{x^2-2x}{x^2+2x}=\lim_{x\to 0}\frac{x(x-2)}{x(x+2)}=\lim_{x\to 0}\frac{x-2}{x+2}=-1.$$

(b) [5 points] $\displaystyle\lim_{x\to -3}\frac{x^2-9}{3+x}$

Solution: $$\lim_{x\to -3}\frac{x^2-9}{3+x}=\lim_{x\to -3}\frac{(x-3)(x+3)}{3+x}=\lim_{x\to -3}(x-3)=-6.$$

3. Find the derivative of each function. You need not simplify the result.

(a) [5 points] $f(x)=3\sqrt[3]{x^2}-\dfrac{2}{\sqrt{x}}$

Solution: $f(x)=3x^{2/3}-2x^{-1/2}$ $\quad f'(x)=2x^{-1/3}+x^{-3/2}$.

(b) [5 points] $f(t)=\dfrac{t^2-1}{t+2}$

Solution: Quotient Rule: $f'(t)=\dfrac{2t(t+2)-1\cdot(t^2-1)}{(t+2)^2}=\dfrac{t^2+4t+1}{(t+2)^2}$.

(c) [5 points] $g(x)=x^2\sqrt{2-x^2}$

Solution: Product Rule: $g'(x)=2x\sqrt{2-x^2}+x^2\dfrac{-2x}{2\sqrt{2-x^2}}=2x\sqrt{2-x^2}-\dfrac{x^3}{\sqrt{2-x^2}}$.

(d) [5 points] $g(t)=e^{3t}\ln(1-2t)$

Solution: Product Rule: $g'(t) = 3e^{3t}\ \ln(1-2t) + e^{3t}\,\dfrac{-2}{1-2t} = 3e^{3t}\ \ln(1-2t) - \dfrac{2e^{3t}}{1-2t}.$

4. [10 points] Use the definition of derivative to find the derivative of $f(x) = x^2$. (No credit will be given when the definition is not used).

Solution: $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{(x+h)^2 - x^2}{h} = \lim\limits_{h\to 0} \dfrac{x^2+2xh+h^2-x^2}{h}$

$= \lim\limits_{h\to 0} \dfrac{2xh+h^2}{h} = \lim\limits_{h\to 0}(2x+h) = 2x.$

5. (a) [5 points] Find $\dfrac{dy}{dx}$ if $x^3y - y^3 = 7$

Solution: $3x^2y + x^3\dfrac{dy}{dx} - 3y^2\dfrac{dy}{dx} = 0, \quad \dfrac{dy}{dx} = \dfrac{3x^2y}{3y^2-x^3}.$

(b) [5 points] Find the equation for the tangent line to the curve $x^3y - y^3 = 7$ at the point $(2, 1)$.

Solution: $m = \dfrac{3\cdot 2^2}{3-2^3} = -\dfrac{12}{5} = -2.4.$

Tangent line equation: $y = 1 - 2.4(x-2), \quad y = -2.4x + 5.8.$

6. For the function $f(x) = \dfrac{8x}{x^2+4}$

(a) [5 points] find horizontal asymptotes

Solution: $\lim\limits_{x\to\pm\infty} f(x) = \lim\limits_{x\to\pm\infty} \dfrac{x}{x^2+4} = 0.$

(b) [5 points] make a sign diagram for the derivative

Solution: $f'(x) = -\dfrac{x^2-4}{(x^2+4)^2},$

$f'(x) > 0$ when $-2 < x < 2$; $f'(x) < 0$ when $x < -2$ or $x > 2$.

(c) [5 points] find all relative maximum and minimum values

Solution: Relative minimum is at $x = -2$, $f(-2) = -\dfrac{1}{4},$

relative maximum is at $x = 2$, $f(2) = \dfrac{1}{4}$.

7. [10 points] Maximum Profit: A furniture store can sell 40 chairs per week at a price of \$80 each. The manager estimates that for each \$5 price reduction she can sell five more chairs per week. The chairs cost the store \$30 each. If x stands for the number of \$5 price reductions, find the price of the chairs and the quantity that maximize the profit. [Hint: Find the price $p(x)$ and the quantity sold $q(x)$ as functions of x. Revenue is price times quantity, cost is the cost per item times quantity, profit is revenue minus cost].

Solution: Price is $p(x) = 80 - 5x$, the quantity sold is $q(x) = 40 + 5x$. Revenue is $R(x) = p(x) \cdot q(x) = (80 - 5x)(40 + 5x) = 3200 + 200x - 25x^2$. Cost is $C(x) = 30q(x) = 1200 + 150x$.

Profit is $P(x) = R(x) - C(x) = 2000 + 50x - 25x^2$.

We maximize the profit by finding its derivative and CNs. $P'(x) = 50 - 50x$.

CNs: $P'(x)$ is defined everywhere, $P'(x) = 0$ when $x = 1$ which is the only CNs of $P(x)$.

$P''(x) = -50 < 0$. Hence profit has a relative maximum at $x = 1$. This maximum is absolute b/c the graph of the profit is a parabola opened down.

$p(1) = 80 - 5 = 75$, $q(1) = 40 + 5 = 45$.

Answer: the price of the chairs that maximize the profit is \$75 and the quantity is 45.

8. [10 points] Two cars start moving from the same point. One travels south at 40 mi/h and the other travels west at 30 mi/h. At what rate is the distance between the cars increasing two hours later?

Solution: Let x be the distance traveled by the first car and y be the distance traveled by the second car; $\dfrac{dx}{dt} = 40$, $\dfrac{dy}{dt} = 30$. The distance between cars is $d(x) = \sqrt{x^2 + y^2}$.

$$\frac{dd}{dt} = \frac{1}{2\sqrt{x^2 + y^2}} \left(2x\frac{dx}{dt} + 2y\frac{dy}{dt} \right) = \frac{1}{\sqrt{x^2 + y^2}} \left(x\frac{dx}{dt} + y\frac{dy}{dt} \right)$$

After two hours $x = 80$ mi, $y = 60$ mi, $d = \sqrt{80^2 + 60^2} = 100$ mi.

Then $\dfrac{dd}{dt} = \dfrac{1}{100} \cdot (80 \cdot 40 + 60 \cdot 30) = 32 + 18 = 50$ mi/h.

9. [10 points] A bank offers 5% compounded continuously. How soon will a deposit triple? Leave answer in exact form.

Solution: $Pe^{0.05t} = 3P$, $e^{0.05t} = 3$, $\ln e^{0.05t} = \ln 3$, $\dfrac{5}{100}t = \ln 3$, $t = \dfrac{100 \ln 3}{5}$,

$t = 20 \ln 3$ years.

10. [10 points] Find the area under the curve $y = xe^{x^2}$ when $0 \le x \le 2$. Leave answer in exact form.

Solution: The area is $A = \int\limits_0^2 xe^{x^2}\,dx$.

Substitution $u = x^2$, $du = 2x\,dx$, $u(0) = 0$, $u(2) = 4$. Then

$$A = \frac{1}{2}\int\limits_0^4 e^u\,du = \frac{1}{2}\,e^u\Big|_0^4 = \frac{e^4 - 1}{2}$$

11. For the demand function $d(x) = 30 - 0.1x^2$ and supply fuction $s(x) = 0.2x^2$ find

(a) [10 points] the market demand level (the positive value of x at which the demand function intersects the supply fuction).

Solution: $30 - 0.1x^2 = 0.2x^2$, $x^2 = 100$, $A = x = 10$.

(b) [10 points] the consumer's surplus at the market demand level found in part (a).

Solution: The market price is $B = s(10) = 20$. Consumer's surplus is

$$\int\limits_0^{10} (d(x) - B)\,dx = \int\limits_0^{10} (10 - 0.1x^2)\,dx = 10x - \frac{1}{30}x^3\Big|_0^{10} = 100 - 33\frac{1}{3} = 66\frac{2}{3}$$

12. [10 points] The population of a town is increasing at the rate of $10t\,e^{t/2}$ people per year, where t is the number of years from now. Find the average gain in population during the next ten years. Leave your answer in exact form.

Solution: The average gain in population during the next ten years is

$$\frac{1}{10}\int\limits_0^{10} 10t\,e^{t/2}\,dt = \int\limits_0^{10} t\,e^{t/2}\,dt$$

By parts: $u = t$, $du = dt$, $dv = e^{t/2}\,dt$, $v = 2e^{t/2}$.

$$= 2t\,e^{t/2}\Big|_0^{10} - \int_0^{10} 2e^{t/2}\,dt = 20e^5 - 4e^{t/2}\Big|_0^{10} = 20e^5 - 4(e^5 - 1) = 16e^5 + 4$$

13. For the function $f(x, y) = e^{x+y}\,\ln x$

(a) [10 points] find the domain

Solution: $\{(x, y)\,|\,x > 0\}$

(b) [10 points] find partials f_y and f_{yx}.

Solution: $f_y = e^{x+y}\,\ln x,\;\; f_{yx} = e^{x+y}\,\ln x + \dfrac{e^{x+y}}{x}.$

14. [10 points] If a company's profit function is

$P(x, y) = 2xy - 2x^2 - 3y^2 + 4x + 18y + 60$ thousand dollars,

find how many of each unit x and y should be produced in order to maximize the profit.

Solution: $P_x = 2y - 4x + 4 = 0 \;\Rightarrow\; y = 2x - 2,\;\; P_y = 2x - 6y + 18 = 0,\;\; x - 3y + 9 = 0,$
$x - 6x + 6 + 9 = 0,\;\; -5x + 15 = 0 \;\Rightarrow\; x = 3,\;\; y = 4.$ CP is $(3, 4)$.

$P_{xx} = -4,\;\; P_{yy} = -6,\;\; P_{xy} = 2,\;\; D = 24 - 4 = 20 > 0,\;\; P_{xx} < 0.$ Relative maximum.

So profit is maximized when $x = 3$, $y = 4$.

15. [10 points] Use Lagrange multipliers to find the maximum value of the function $f(x, y) = xy$ subject to the constraint $2x + y = 12$.

[Hint: Find CP. The maximum value of the function is attained at CP.]

Solution: $F(x, y, \lambda) = xy + 2x\lambda + y\lambda - 12\lambda$

$F_x = y + 2\lambda = 0,\;\; F_y = x + \lambda = 0,\;\; F_\lambda = 2x + y - 12 = 0$

$x = -\lambda,\;\; y = -2\lambda,\quad -2\lambda - 2\lambda - 12 = 0,\;\; \lambda = -3 \;\Rightarrow\; x = 3,\;\; y = 6.$

CP is $(3, 6)$.

The maximum value of the function is $f(3, 6) = 18$.