Practice Problems to Midterm Exam 1

Complete the following questions. Show all your work (no work = no credit). Write neatly. Simplify your answers when possible.

Sections 5.5, 6.1, and 6.2 **Integration**

1. (10 points) Identify any errors in each solution. Clearly explain the mistakes and how you could correct them.

   (a) (5 points) **Problem Statement:** Evaluate the integral $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin t}{\cos t}\, dt$.

   Solution: Let $u = \sin t$, then $du = \cos t$.

   Substituting, $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin t}{\cos t}\, dt = \int_0^{\frac{\pi}{4}} u\, du = \frac{u^2}{2} + C.$

   (b) (5 points) **Problem Statement:** Evaluate the integral $\displaystyle\int t \ln t\, dt$

   Solution: Letting $u = t$ and $dv = \ln t$, then $du = 1$ and $v = \dfrac{1}{t}$.

   Integration by parts gives, $\displaystyle\int t \ln t\, dt = t\left(\frac{1}{t}\right) + \int \frac{1}{t}\, dt = 1 + \ln t.$

2. (10 points) Evaluate the integral $\displaystyle I = \int x\sqrt{4 - x^2}\, dx$

3. (10 points) Evaluate the integral $\displaystyle I = \int \sqrt{4 - x^2}\, dx$

4. Find $\int \sin(\theta)\cos(\theta)\, d\theta$ (1)

   (a) You probably solved (1) using the substitution $u = \sin(\theta)$ or $u = \cos(\theta)$.
   Now find $\int \sin(\theta)\cos(\theta)\, d\theta$ using the <u>other</u> substitution. (i.e. the one you did not use in (1)).

   (b) There is another way of finding this integral which involves the trig identities

   $$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$
   $$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$

   Find $\int \sin(\theta)\cos(\theta)\, d\theta$ using one of these identities and the substitution $u = 2\theta$.

   (c) You should now have three different expressions for the indefinite integral $\int \sin(\theta)\cos(\theta)\, d\theta$.

   Are they really different? Are they all correct? Explain.

Section 6.3 **Partial fractions**

5. (a) (5 points) Find the partial fraction decomposition of $\dfrac{x+2}{x^2-x}$.

   (b) (5 points) Evaluate the integral $I = \displaystyle\int_2^3 \frac{x+2}{x^2-x}\,dx$. Simplify your answer.

Section 6.6 **Improper Integrals**

6. (10 points) Write correct formula for evaluating the improper integral $I = \displaystyle\int_0^\infty \frac{3}{e^x\sqrt{x-2}}\,dx$

   Do not integrate!

7. (10 points) Evaluate the improper integral $I = \displaystyle\int_4^5 \frac{6x}{\sqrt{x^2-16}}\,dx$

   if it is convergent or show that it is divergent.

8. (10 points) Mike was asked to evaluate the integral $I = \displaystyle\int_0^{10} \frac{a+1}{x^2-(a-1)x-a}\,dx$

   Below is his solution:

$$\frac{a+1}{x^2-(a-1)x-a} = \frac{a+1}{(x-a)(x+1)} = \frac{1}{x-a} - \frac{1}{x+1}$$

$$I = \int_0^{10} \frac{1}{x-a}\,dx - \int_0^{10} \frac{1}{x+1}\,dx = \ln|x-a|\Big|_0^{10} - \ln|x+1|\Big|_0^{10}$$

$$I = \ln|10-a| - \ln|a| - \ln 11 + \ln 1 = \ln\frac{|10-a|}{11|a|}$$

   Explain why Mike's solution cannot be considered as a right one. Correct his answer. Do not integrate!

Section 7.1 **Areas Between Curves**

9. (10 points) Consider the region R bounded by parabola $y = -x^2 + 4x$ with vertex at $(2, 4)$ and by lines $x = 0,\ x = 3,\ y = ax + 4$, where $a > 0$. The area of R is 12. Find the value of a. Support your answer.

Sections 7.2 and 7.3 **Volumes**

10. (10 points) Consider the region R bounded by lines $y = 0,\ x = 1$, and the function $f(x) = a\sqrt{x}$, where $a > 0$. The volume V_1 is generated by rotating the region R about the axis $x = 0$. The volume V_2 is generated by rotating the region R about the axis $y = 0$. Find the value of a for which $V_1 = V_2$. Support your answer.

11. (10 points) By using the method of washers find the volume of the solid generated by rotating the region bounded by the curves $y = \sqrt{x}$ and $y = x^2$ about the line $y = 1$.

12. (10 points) Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the curves $y = x^2$, $y = 2x$ about the axis $x = 3$.

Section 7.4 **Arc Length**

13. (10 points) Mary was asked to evaluate arc length of the curve $y = \sqrt{|x|}$ when x changes from -2 to 3. She found that the arc length L can be evaluated by using the integral

$$L = \int_{-2}^{3} \sqrt{1 + \frac{1}{2x}}\, dx$$

What is wrong in Mary's formula? Correct her answer. Do not integrate!

14. (10 points) Find the exact length L of the curve $y = \dfrac{x^2}{4} - \dfrac{\ln x}{2}$ when $1 \le x \le 2$.

Hint: $1 + \left(\dfrac{x}{2} - \dfrac{1}{2x}\right)^2 = 1 + \left(\dfrac{x}{2}\right)^2 - \dfrac{1}{2} + \left(\dfrac{1}{2x}\right)^2 = \left(\dfrac{x}{2}\right)^2 + \dfrac{1}{2} + \left(\dfrac{1}{2x}\right)^2 = \left(\dfrac{x}{2} + \dfrac{1}{2x}\right)^2.$

Section 7.6 **Applications to physics and engineering (work, force)**

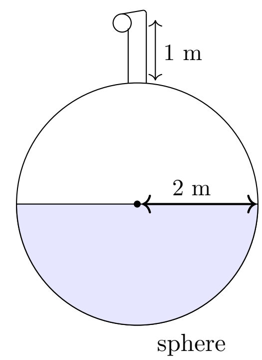
15. (10 points) A chain laying on the ground is 12 m long and its mass is 72 kg. How much work is required to raise one end of the chain to the height of 8 m?

16. (10 points) A cylindrical aquarium has radius of 4 feet and height of 7 feet. The depth of water is 4 feet. How much work is required to pump all of the water out over the side?

    (You may use the following equality $62.5 = \dfrac{1000}{16}$ lb/ft^2)

17. (10 points) A water tank has the shape of a sphere of radius 2 m with a spout at the top of length of 1 m, as in the picture below. The spherical part of the tank is half full of water and the spout is empty.

    Set up **but do not evaluate** an integral to find the work required to empty the tank by pumping all of the water out of the spout. The density of water is $\rho = 1000\frac{\text{kg}}{\text{m}^3}$ and the acceleration due to gravity is $g = 9.8\frac{\text{m}}{\text{s}^2}$.



Section 7.7 **Differential Equations and Applications of DEs**

18. Consider the initial value problem:

$$y' - (\tan x)\, y = 1, \qquad y(0) = -3.$$

(a) (2 points) Determine the type of the differential equation.

(b) (6 points) Find the general solution of the problem.

(c) (2 points) Find the particular solution of this problem that satisfies given initial condition.

19. Determine the type of the given differential equation. Solve the initial-value problem. Show all work.

(a) (10 points) $xy' = (1+x)y, \qquad y(1) = e.$

(b) (10 points) $y' + \dfrac{2x}{x^2+1}y = \dfrac{1}{x^2+1}, \qquad y(0) = 3.$

20. A large tank initially contains 10 liters of salty water with a concentration of 60 grams of salt per liter of water. At time zero, a solution with concentration of 10 grams of salt per liter begins flowing into the tank at a rate of 2 liters per minute. Simultaneously, the well mixed solution drains out at the rate of 2 liters per minute.

(a) (1 point) Find the initial condition.

(b) (2 points) Write down the balance equation that models the process.

(c) (6 points) Solve the balance equation to find a formula of the amount of salt in the tank after t minutes?

(d) (1 points) How much salt is in the tank after 10 minutes? Leave answer in exact form.

Section **Inhomogeneous Second Order Differential Equations**

21. (10 points) Consider the second-order ordinary differential equation for $y = y(x)$:

$$3y'' - 6y' + ky = G(x),$$

where k is a constant, and $G(x)$ is a function of x.

(a) (5 points) Find the value of k such that the corresponding characteristic equation has two repeated real roots, and then for this value of k find the general solution to the corresponding homogeneous differential equation.

(b) (5 points) If $G(x) = P_m(x)\cos x$ where $P_m(x)$ is a polynomial of degree $m > 0$, what is the trial form of a particular solution to the inhomogeneous differential equation? For $m = 1$ pick an example of $P_m(x)$ and find a form of a particular solution to the inhomogeneous equation.

22. (10 points) For the equation $y'' - 4y' - 5y = 12e^{-t}$.

(a) (3 points) Find the solution of the corresponding homogeneous equation.

(b) (5 points) Find a particular solution by using the method of undetermined coefficients.

(c) (2 points) Find the general solution.

Section **Oscillations**

23. (10 points) A forced mass spring system with an external driving force is modeled by

$$x'' + 4x' + 5x = 10\sin 3t,$$

where t is measured in seconds and x in meters.

(a) (3 points) Find the transient state, i.e. the solution to the associated homogeneous equation.

(b) (5 points) By using the method of undetermined coefficients find the steady-state, i.e. a particular solution, to the forced equation.

(c) (2 points) Write down the general solution.