HIGH ACCURACY METHOD FOR MAGNETOHYDRODYNAMICS SYSTEM IN ELSÄSSER VARIABLES

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Abstract. The MHD flows are governed by the Navier-Stokes equations coupled with the Maxwell equations through coupling terms. We prove the unconditional stability of a partitioned method for the evolutionary full MHD equations, at *high* magnetic Reynolds number, in the Elsässer variables. The method we propose is a defect correction second order scheme, and entails the implicit discretization of the subproblem terms and the explicit discretization of coupling terms.

1. Introduction. The equations of magnetohydrodynamics (MHD) describe the motion of electrically conducting, incompressible flows in the presence of a magnetic field. When an electrically conducting fluid moves in a magnetic field, the magnetic field exerts forces which may substantially modify the flow. Conversely, the flow itself gives rise to a second, induced field and thus modifies the magnetic field. Initiated by Alfvèn in 1942 [1], MHD is widely exploited in numerous branches of science including astrophysics and geophysics [24, 35, 16, 12, 11, 3, 6, 15], as well as engineering, e.g., liquid metal cooling of nuclear reactors [2, 22, 38], process metallurgy [8], sea water propulsion [31].

The MHD flows entails two distinct physical processes: the motion of fluid is governed by hydrodynamics equations and the magnetic field is governed by Maxwell equations. One approach to solve the coupled problem is by monolithic methods, or implicit (fully coupled) algorithms, that are robust and stable, but quite demanding in computational time and resources. In these methods, the globally coupled problem is assembled at each time step and then solved iteratively. Partitioned methods, which solve the coupled problem by successively solving the sub-physics problems [30], are another attractive and promising approach for solving MHD system.

Most terrestrial applications, in particular most industrial and laboratory flows, involve *small* magnetic Reynolds number. In this cases, while the magnetic field considerably alters the fluid motion, the induced field is usually found to be negligible by comparison with the imposed field [28, 36, 8]. Neglecting the induced magnetic field one can reduce the MHD systems to the significantly simpler Reduced MHD (RMHD), for which several implicit-explicit (IMEX) schemes were studied in [29].

In this report we aim to improve the accuracy of the first order method introduced in [41]. The method that we aim to develop for the evolutionary full MHD equations, at high magnetic Reynolds number in the Elsässer variables, also needs to be stable and allow for explicit-implicit implementations with different time scales.

To that end, we employ the spectral deferred correction (SDC) method, proposed for stiff ODEs by Dutt et al., [13], and further developed by Minion et al.; see [33, 34, 7] and the references therein. SDC methods were studied and compared to intrinsically high-order methods such as additive Runge-Kutta methods and linear

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multistep methods based on BDFs, with the conclusion that the SDC methods are at least comparable to the latter. In addition, achieving high accuracy for the turbulent NSE using Runge-Kutta-based methods is very expensive, and the BDF-based methods typically do not perform well in problems where relevant time scales associated with different terms in the equation are widely different; see, e.g., [7] for an example of an advection-diffusion-reaction problem for which the SDC is the best choice for high-accuracy temporal discretization.

The equations of magnetohydrodynamics describing the motion of an incompressible fluid flow in presence of a magnetic field are the following (see, e.g. [28, 4, 5])

$$\begin{aligned} &\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - (\boldsymbol{B}\cdot\nabla)\boldsymbol{B} - \nu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f}, && \nabla\cdot\boldsymbol{u} = 0,\\ &\frac{\partial \boldsymbol{B}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{B} - (\boldsymbol{B}\cdot\nabla)\boldsymbol{u} - \nu_m\Delta\boldsymbol{B} = \nabla\times\boldsymbol{g}, && \nabla\cdot\boldsymbol{B} = 0, \end{aligned}$$

in $\Omega\times(0,T)$, where Ω is the fluid domain, $\boldsymbol{u} = (u_1(\boldsymbol{x},t), u_2(\boldsymbol{x},t), u_3(\boldsymbol{x},t))$ is the fluid velocity, $p(\boldsymbol{x},t)$ is the pressure, $\boldsymbol{B} = (B_1(\boldsymbol{x},t), B_2(\boldsymbol{x},t), B_3(\boldsymbol{x},t))$ is the magnetic field, $\boldsymbol{f}$ and $\nabla\times\boldsymbol{g}$ are external forces, ν is the kinematic viscosity and ν_m is the magnetic resistivity. The total magnetic field can be split in two parts $\boldsymbol{B} = \boldsymbol{B}_\circ + \boldsymbol{b}$ (mean and fluctuations). We prescribe homogeneous Dirichlet boundary conditions for $\boldsymbol{u}$, and $\boldsymbol{B} = \boldsymbol{B}_\circ$ on the boundary (see [18] for typical magnetic boundary conditions).

Then the Elsässer fields [14]

$$\boldsymbol{z}^+ = \boldsymbol{u} + \boldsymbol{b}, \quad \boldsymbol{z}^- = \boldsymbol{u} - \boldsymbol{b}, \tag{1.1}$$

merging the physical properties of the Navier-Stokes and Maxwell equations, suggest stable time-splitting schemes for the full MHD equations. The momentum equations, in the Elsässer variables, are

$$\frac{\partial \boldsymbol{z}^\pm}{\partial t} \mp (\boldsymbol{B}_\circ\cdot\nabla)\boldsymbol{z}^\pm + (\boldsymbol{z}^\mp\cdot\nabla)\boldsymbol{z}^\pm - \frac{\nu+\nu_m}{2}\Delta\boldsymbol{z}^\pm - \frac{\nu-\nu_m}{2}\Delta\boldsymbol{z}^\mp + \nabla p = f^\pm, \tag{1.2}$$

while the continuity equations are $\nabla\cdot\boldsymbol{z}^\pm = 0$. We note that the nonlinear interactions occur between the Alfvènic fluctuations $\boldsymbol{z}^\pm$. The mean magnetic field plays an important role in MHD turbulence, for example it can make the turbulence anisotropic; suppress the turbulence by decreasing energy cascade, etc. In the presence of a strong mean magnetic field, $\boldsymbol{z}^+$ and $\boldsymbol{z}^-$ wavepackets travel in opposite directions with the phase velocity of $\boldsymbol{B}_\circ$, and interact weakly. For Kolmogorov's and Iroshnikov/Kraichnan's phenomenological theories of MHD isotropic and anisotropic turbulence, see [25, 27, 9, 32, 42, 37, 17, 20, 43].

In a "classical" understanding, the deferred correction approach to solving ODEs is based on replacing the original ODE (in our case, the system of ODEs obtained from the original PDEs by the Method of Lines) with the corresponding Picard integral equation, discretizing the time interval, solving the integral equation approximately and then correcting the solution by solving a sequence of error equations on the same grid with the same scheme; see [13] and [33] for the detailed mathematical presentation of SDC. In particular, the two-step Deferred Correction method introduced in this paper, performs as follows. The first approximation to the sought quantities (in this case, the Elsässer variables $\boldsymbol{z}^+, \boldsymbol{z}^-$) is obtained by the stable and computationally attractive first order accurate IMEX method of [41]. Then the second order accurate approximation is computed, which improves the accuracy without sacrificing stability.

Note that the second step utilizes the same IMEX time discretization as in the first step; only the right-hand side is modified by a known quantity, i.e, a known solution from the first step. This results in the computational attractiveness of the method: computing two low-order accurate approximations is much less costly (especially for very stiff problems) than computing a single higher-order approximation.

2. Notation and Preliminaries. We consider a domain $\Omega \subset \mathbb{R}^d$ (d=2 or 3) to be a convex polygon or polyhedra. We denote the familiar Lesbegue measure spaces by $L^p(\Omega)$, and denote the $L^2(\Omega)$ inner product and induced norm by $(\cdot,\cdot)$ and $\|\cdot\|$ respectively. Additionally, we denote the $L^\infty(\Omega)$ norm by $\|\cdot\|_\infty$, and the norm associated with the Sobolev spaces $W^{2,k}(\Omega) \equiv H^k(\Omega)$ by $\|\cdot\|_k$. All other norms will be clearly labeled.

Throughout the article, we will make use of the inequalities presented in the following lemma. In Lemma 2.1 and subsequent analysis we denote constants that are independent of ν and ν_m by C. The generic constant varies throughout this work.

LEMMA 2.1. *If* $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in H^1(\Omega)$, *and* $\nabla\cdot\boldsymbol{u}=0$ *then*

$$
\begin{aligned}
&((\boldsymbol{u}\cdot\nabla)\boldsymbol{v},\boldsymbol{w}) = -((\boldsymbol{u}\cdot\nabla)\boldsymbol{w},\boldsymbol{v}),\\
&((\boldsymbol{u}\cdot\nabla)\boldsymbol{v},\boldsymbol{v}) = 0,\\
&((\boldsymbol{u}\cdot\nabla)\boldsymbol{v},\boldsymbol{w}) \le C\|\boldsymbol{u}\|\|\nabla\boldsymbol{v}\|_\infty\|\boldsymbol{w}\| \ \textit{(provided } \nabla\boldsymbol{v}\in L^\infty),\\
&((\boldsymbol{u}\cdot\nabla)\boldsymbol{v},\boldsymbol{w}) \le C\|\boldsymbol{u}\|_\infty\|\nabla\boldsymbol{v}\|\|\boldsymbol{w}\| \ \textit{(provided } \boldsymbol{u}\in L^\infty),\\
&((\boldsymbol{u}\cdot\nabla)\boldsymbol{v},\boldsymbol{w}) \le C\|\boldsymbol{u}\|^{\frac12}\|\nabla\boldsymbol{u}\|^{\frac12}\|\nabla\boldsymbol{v}\|\|\nabla\boldsymbol{w}\|, \ \textit{and}\\
&((\boldsymbol{u}\cdot\nabla)\boldsymbol{v},\boldsymbol{w}) \le C\|\boldsymbol{u}\|^{\frac14}\|\nabla\boldsymbol{u}\|^{\frac34}\|\nabla\boldsymbol{v}\|\|\boldsymbol{w}\|^{\frac14}\|\nabla\boldsymbol{w}\|^{\frac34}.
\end{aligned}
$$

These are classical results used in the study of Navier-Stokes equations and magnetohydrodynamics (see for example [39, 40, 10]). In addition to the above results we will also employ the discrete Gronwall's lemma [23].

3. The implicit-explicit partitioned schemes. The heart of any partitioned method, aiming at decoupling the two physically interconnected subproblems, is its treatment of the coupling terms. The method we study herein has the coupling terms lagged or extrapolated in a careful way that preserves stability.

3.1. First order unconditionally stable IMEX partitioned scheme. The method proposed and analyzed in [41] has the coupling terms lagged, thus the system uncouples into two subproblem solves. It approximates the momentum equations (1.2) and continuity equations in the Elsässer variables by the following first-order IMEX scheme (backward-Euler forward-Euler)

$$
\begin{aligned}
&\frac{{}_0\boldsymbol{z}^{\pm}_{n+1} - {}_0\,\boldsymbol{z}^{\pm}_{n}}{\Delta t} \mp (\boldsymbol{B}_\circ\cdot\nabla){}_0\boldsymbol{z}^{\pm}_{n+1} + ({}_0\boldsymbol{z}^{\mp}_{n}\cdot\nabla){}_0\boldsymbol{z}^{\pm}_{n+1} \qquad (3.1)\\
&\qquad\qquad - \frac{\nu+\nu_m}{2}\Delta_0\boldsymbol{z}^{\pm}_{n+1} - \frac{\nu-\nu_m}{2}\Delta_0\boldsymbol{z}^{\mp}_{n} + \nabla_0 p^{\pm}_{n+1} = \boldsymbol{f}^{\pm}(t_{n+1}),\\
&\nabla\cdot\boldsymbol{z}^{\pm}_{n+1} = 0. \qquad (3.2)
\end{aligned}
$$

The scheme (3.1)-(3.2) is modular, i.e., the variables ${}_0\boldsymbol{z}^{+}$ and ${}_0\boldsymbol{z}^{-}$ are decoupled, and is unconditionally absolute-stable. We note that the pre-subscript occurring on the variables ${}_0\boldsymbol{z}^{\pm}_n$ is used to denote the first order IMEX approximation to the Elsässer variables $\boldsymbol{z}(t_n)^{\pm}$ respectively.

As mentioned before the unconditional stability of the IMEX method is proven in [41]. Thus, we restrict our attention to showing the method is first order accurate in time. We note the term η_n occurs regularly in the analysis below. The value of η_n is between the timesteps t_n and t_{n+1} the arises from our use of Taylor series. So, it is unknown and varies in each occurrence.

LEMMA 3.1. *Given a final time $T > 0$ and timestep $\Delta t > 0$ let $t_n = n \times \Delta t$ for $n = 0, 1, ..., N$. Let ${}_0\boldsymbol{z}_n^{\pm}$ denote the IMEX approximation of $\boldsymbol{z}^{\pm}(t^n)$, and let ${}_0\boldsymbol{e}_n^{\pm} := \boldsymbol{z}^{\pm}(t_n) - {}_0\,\boldsymbol{z}_n^{\pm}$. Then provided the true solution satisfies the regularity assumptions*

$$\nabla \boldsymbol{z}^{\pm} \in L^2((0,T); L^\infty(\Omega)), \quad \partial_t \boldsymbol{z}^{\pm} \in L^2((0,T); L^2(\Omega)),$$
$$\partial_{tt} \boldsymbol{z}^{\pm} \in L^2((0,T); L^2(\Omega)), \quad \nabla \partial_t \boldsymbol{z}^{\pm} \in L^2((0,T); L^2(\Omega)),$$

the following error estimate holds

$$\begin{aligned}
&\|{}_0\boldsymbol{e}_N^+\|^2 + \|{}_0\boldsymbol{e}_N^-\|^2 + \Delta t \frac{\nu\nu_m}{2(\nu+\nu_m)} \sum_{n=0}^{N} (\|\nabla {}_0\boldsymbol{e}_n^+\|^2 + \|\nabla {}_0\boldsymbol{e}_n^-\|^2) \\
&\leq \Delta t^2 C \frac{\nu+\nu_m}{\nu\nu_m} \exp\left(\Delta t \sum_{n=0}^{N-1} (\|\nabla \boldsymbol{z}^-(t_n)\|^2 + \|\nabla \boldsymbol{z}^+(t_n)\|^2 \right) \times \\
&\quad \times \Delta t \sum_{n=0}^{N} \Big(\|\partial_{tt} \boldsymbol{z}^+(\eta_n)\|^2 + \|\partial_t \boldsymbol{z}^-(\eta_n)\|^2 \|\nabla \boldsymbol{z}^+(t_n)\|_\infty^2 \\
&\qquad + (\nu_m - \nu)^2 (\|\nabla \partial_t \boldsymbol{z}^-(\eta_n)\|^2 + \|\nabla \partial_t \boldsymbol{z}^+(\eta_n)\|^2) \\
&\qquad + \|\partial_{tt} \boldsymbol{z}^-(\eta_n)\|^2 + \|\partial_t \boldsymbol{z}^+(\eta_n)\|^2 \|\nabla \boldsymbol{z}^-(t_n)\|_\infty^2 \Big).
\end{aligned}$$

Proof. We begin by rewriting the first continuous momentum equation as

$$\begin{aligned}
&\frac{1}{\Delta t}(\boldsymbol{z}^+(t_{n+1}) - \boldsymbol{z}^+(t_n)) + (\boldsymbol{z}^-(t_n)\cdot\nabla)\boldsymbol{z}^+(t_{n+1}) - (\boldsymbol{B}_\circ \cdot \nabla)\boldsymbol{z}^+(t_{n+1}) \\
&\quad - \frac{\nu+\nu_m}{2}\Delta \boldsymbol{z}^+(t_{n+1}) - \frac{\nu-\nu_m}{2}\Delta \boldsymbol{z}^-(t_n) + \nabla p(t_{n+1}) \\
&= \frac{1}{\Delta t}(\boldsymbol{z}^+(t_{n+1}) - \boldsymbol{z}^+(t_n)) - \partial_t \boldsymbol{z}^+(t_{n+1}) + \frac{\nu-\nu_m}{2}\Delta \boldsymbol{z}^-(t_{n+1}) - \frac{\nu-\nu_m}{2}\Delta \boldsymbol{z}^-(t_n) \\
&\quad + (\boldsymbol{z}^-(t_n)\cdot\nabla)\boldsymbol{z}^+(t_{n+1}) - (\boldsymbol{z}^-(t_{n+1})\cdot\nabla)\boldsymbol{z}^+(t_{n+1}) + \boldsymbol{f}^+(t_{n+1}).
\end{aligned} \tag{3.3}$$

The first momentum equation for the IMEX method is

$$\begin{aligned}
&\frac{1}{\Delta t}({}_0\boldsymbol{z}_{n+1}^+ - {}_0\,\boldsymbol{z}_n^+) + ({}_0\boldsymbol{z}_n^- \cdot \nabla){}_0\boldsymbol{z}_{n+1}^+ - (\boldsymbol{B}_\circ \cdot \nabla){}_0\boldsymbol{z}_{n+1}^+ - \frac{\nu+\nu_m}{2}\Delta_0 \boldsymbol{z}_{n+1}^+ \\
&\quad - \frac{\nu-\nu_m}{2}\Delta_0 \boldsymbol{z}_n^- + \nabla_0 p_{n+1}^+ = \boldsymbol{f}^+(t_{n+1}).
\end{aligned} \tag{3.4}$$

The second continuous momentum equation may be expressed as

$$\begin{aligned}
&\frac{1}{\Delta t}(\boldsymbol{z}^-(t_{n+1}) - \boldsymbol{z}^-(t_n)) + (\boldsymbol{z}^+(t_n)\cdot\nabla)\boldsymbol{z}^-(t_{n+1}) + (\boldsymbol{B}_\circ \cdot \nabla)\boldsymbol{z}^-(t_{n+1}) \\
&\quad - \frac{\nu+\nu_m}{2}\Delta \boldsymbol{z}^-(t_{n+1}) - \frac{\nu-\nu_m}{2}\Delta \boldsymbol{z}^+(t_n) \\
&= \frac{1}{\Delta t}(\boldsymbol{z}^-(t_{n+1}) - \boldsymbol{z}^-(t_n)) - \partial_t \boldsymbol{z}^-(t_{n+1}) - \nabla p(t_{n+1}) + (\boldsymbol{z}^+(t_n)\cdot\nabla)\boldsymbol{z}^-(t_{n+1})
\end{aligned} \tag{3.5}$$

$$- (\boldsymbol{z}^+(t_{n+1})\cdot\nabla)\boldsymbol{z}^-(t_{n+1}) + \tfrac{\nu-\nu_m}{2}\Delta\boldsymbol{z}^-(t_{n+1}) - \tfrac{\nu-\nu_m}{2}\Delta\boldsymbol{z}^+(t_n) + \boldsymbol{f}^-(t_{n+1}).$$

The second momentum equation for the IMEX method satisfies

$$\begin{aligned}\frac{1}{\Delta t}({}_0\boldsymbol{z}^-_{n+1} - {}_0\,\boldsymbol{z}^-_n) + ({}_0\boldsymbol{z}^+_n\cdot\nabla){}_0\boldsymbol{z}^-_{n+1} + (\boldsymbol{B}_\circ\cdot\nabla){}_0\boldsymbol{z}^-_{n+1} \\ - \frac{\nu+\nu_m}{2}\Delta_0\boldsymbol{z}^-_{n+1} - \frac{\nu-\nu_m}{2}\Delta_0\boldsymbol{z}^+_n + \nabla_0 p^-_{n+1} = \boldsymbol{f}^-(t_{n+1}).\end{aligned} \tag{3.6}$$

Subtracting (3.4) from (3.3) and multiplying by ${}_0\boldsymbol{e}^+_{n+1}$, subtracting (3.6) from (3.5) and multiplying by ${}_0\boldsymbol{e}^-_{n+1}$, adding the resulting equations and reducing gives

$$\begin{aligned}&\frac{1}{2\Delta t}(\|{}_0\boldsymbol{e}^+_{n+1}\|^2 - \|{}_0\boldsymbol{e}^+_n\|^2) + \frac{1}{2\Delta t}(\|{}_0\boldsymbol{e}^-_{n+1}\|^2 - \|{}_0\boldsymbol{e}^-_n\|^2) \\ &\quad + \frac{(\nu-\nu_m)^2}{4(\nu+\nu_m)}(\|\nabla_0\boldsymbol{e}^+_{n+1}\|^2 - \|\nabla_0\boldsymbol{e}^+_n\|^2 + \|\nabla_0\boldsymbol{e}^-_{n+1}\|^2 - \|\nabla_0\boldsymbol{e}^-_n\|^2) \\ &\quad + \frac{\nu\nu_m}{\nu+\nu_m}(\|\nabla_0\boldsymbol{e}^+_{n+1}\|^2 + \|\nabla_0\boldsymbol{e}^-_{n+1}\|^2) \\ &\le (({}_0\boldsymbol{z}^-_n\cdot\nabla){}_0\boldsymbol{z}^+_{n+1}, {}_0\boldsymbol{e}^+_{n+1}) - ((\boldsymbol{z}^-(t_n)\cdot\nabla)\boldsymbol{z}^+(t_{n+1}), {}_0\boldsymbol{e}^+_{n+1}) \\ &\quad + ((\boldsymbol{z}^+(t_n)\cdot\nabla)\boldsymbol{z}^-(t_{n+1}), {}_0\boldsymbol{e}^-_{n+1}) - ((\boldsymbol{z}^+(t_{n+1})\cdot\nabla)\boldsymbol{z}^-(t_{n+1}), {}_0\boldsymbol{e}^-_{n+1}) \\ &\quad + (({}_0\boldsymbol{z}^+_n\cdot\nabla){}_0\boldsymbol{z}^-_{n+1}, {}_0\boldsymbol{e}^-_{n+1}) - ((\boldsymbol{z}^+(t_n)\cdot\nabla)\boldsymbol{z}^-(t_{n+1}), {}_0\boldsymbol{e}^-_{n+1}) \\ &\quad + ((\boldsymbol{z}^-(t_n)\cdot\nabla)\boldsymbol{z}^+(t_{n+1}), {}_0\boldsymbol{e}^+_{n+1}) - ((\boldsymbol{z}^-(t_{n+1})\cdot\nabla)\boldsymbol{z}^+(t_{n+1}), {}_0\boldsymbol{e}^+_{n+1}) \\ &\quad + (\frac{\boldsymbol{z}^-(t_{n+1}) - \boldsymbol{z}^-(t_n)}{\Delta t} - \partial_t\boldsymbol{z}^+(t_{n+1}), {}_0\boldsymbol{e}^-_{n+1}) \\ &\quad + (\frac{\boldsymbol{z}^+(t_{n+1}) - \boldsymbol{z}^+(t_n)}{\Delta t} - \partial_t\boldsymbol{z}^+(t_{n+1}), {}_0\boldsymbol{e}^+_{n+1}) \\ &\quad + \frac{\nu_m-\nu}{2}\big(\nabla(\boldsymbol{z}^-(t_{n+1}) - \boldsymbol{z}^-(t_n)), \nabla_0\boldsymbol{e}^+_{n+1}\big) \\ &\quad + \frac{\nu_m-\nu}{2}\big(\nabla(\boldsymbol{z}^+(t_{n+1}) - \boldsymbol{z}^+(t_n)), \nabla_0\boldsymbol{e}^-_{n+1}\big).\end{aligned} \tag{3.7}$$

The proof continues by bounding the terms occurring on the RHS of (3.7)

$$\begin{aligned}&(({}_0\boldsymbol{z}^-_n\cdot\nabla){}_0\boldsymbol{z}^+_{n+1}, {}_0\boldsymbol{e}^+_{n+1}) - ((\boldsymbol{z}^-(t_n)\cdot\nabla)\boldsymbol{z}^+(t_{n+1}), {}_0\boldsymbol{e}^+_{n+1}) \\ &= (({}_0\boldsymbol{e}^+_n)\cdot\nabla)\boldsymbol{z}^+(t_{n+1}), {}_0\boldsymbol{e}^+_{n+1}) \le C\gamma^{-1}\|{}_0\boldsymbol{e}^+_n\|^2\|\nabla\boldsymbol{z}^+(t_{n+1})\|^2_\infty + \gamma\|\nabla_0\boldsymbol{e}^+_{n+1}\|^2,\end{aligned} \tag{3.8}$$

$$\begin{aligned}&((\boldsymbol{z}^+(t_n)\cdot\nabla)\boldsymbol{z}^-(t_{n+1}), {}_0\boldsymbol{e}^-_{n+1}) - ((\boldsymbol{z}^+(t_{n+1})\cdot\nabla)\boldsymbol{z}^-(t_{n+1}), {}_0\boldsymbol{e}^-_{n+1}) \\ &\quad = |\big(((\boldsymbol{z}^+(t_n) - \boldsymbol{z}^+(t_{n+1}))\cdot\nabla)\boldsymbol{z}^-(t_{n+1}), {}_0\boldsymbol{e}^-_{n+1}\big)| \\ &\quad \le C\Delta t^2\gamma^{-1}\|\partial_t\boldsymbol{z}^+(\eta_n)\|^2\|\nabla\boldsymbol{z}^-(t_{n+1})\|^2_\infty + \gamma\|\nabla_0\boldsymbol{e}^-_{n+1}\|^2,\end{aligned} \tag{3.9}$$

$$\begin{aligned}&(({}_0\boldsymbol{z}^+_n\cdot\nabla){}_0\boldsymbol{z}^-_{n+1}, {}_0\boldsymbol{e}^-_{n+1}) - ((\boldsymbol{z}^+(t_n)\cdot\nabla)\boldsymbol{z}^-(t_{n+1}), {}_0\boldsymbol{e}^-_{n+1}) \\ &\quad = (({}_0\boldsymbol{e}^+_n\cdot\nabla)\boldsymbol{z}^-(t_{n+1}), {}_0\boldsymbol{e}^-_{n+1}) \le C\gamma^{-1}\|{}_0\boldsymbol{e}^+_n\|^2\|\nabla\boldsymbol{z}^-(t_{n+1})\|^2_\infty + \gamma\|\nabla_0\boldsymbol{e}^-_{n+1}\|^2,\end{aligned} \tag{3.10}$$

$$\begin{aligned}&((\boldsymbol{z}^-(t_n)\cdot\nabla)\boldsymbol{z}^+(t_{n+1}), {}_0\boldsymbol{e}^+_{n+1}) - ((\boldsymbol{z}^-(t_{n+1})\cdot\nabla)\boldsymbol{z}^+(t_{n+1}), {}_0\boldsymbol{e}^+_{n+1}) \\ &\quad = \big(((\boldsymbol{z}^-(t_n) - \boldsymbol{z}^-(t_{n+1}))\cdot\nabla)\boldsymbol{z}^+(t_{n+1}), {}_0\boldsymbol{e}^+_{n+1}\big) \\ &\quad \le C\gamma^{-1}\Delta t^2\|\partial_t\boldsymbol{z}^-(\eta_n)\|^2\|\nabla\boldsymbol{z}^+(t_n)\|^2_\infty + \gamma\|\nabla_0\boldsymbol{e}^+_{n+1}\|^2,\end{aligned} \tag{3.11}$$

$$\begin{aligned}&(\frac{\boldsymbol{z}^-(t_{n+1}) - \boldsymbol{z}^-(t_n)}{\Delta t} - \partial_t\boldsymbol{z}^+(t_{n+1}), {}_0\boldsymbol{e}^-_{n+1}) \\ &\quad \le C\Delta t^2\gamma^{-1}\|\partial_{tt}\boldsymbol{z}^-(\eta_n)\|^2 + \gamma\|\nabla_0\boldsymbol{e}^-_{n+1}\|^2,\end{aligned} \tag{3.12}$$

$$
\begin{aligned}
&(\frac{\boldsymbol{z}^+(t_{n+1})-\boldsymbol{z}^+(t_n)}{\Delta t}-\partial_t\boldsymbol{z}^+(t_{n+1}),{}_0\boldsymbol{e}^+_{n+1}) && (3.13)\\
&\quad\le C\Delta t^2\gamma^{-1}\|\partial_{tt}\boldsymbol{z}^+(\eta_n)\|^2+\gamma\|\nabla_0\boldsymbol{e}^+_{n+1}\|^2,\\
&\frac{\nu_m-\nu}{2}\big(\nabla(\boldsymbol{z}^-(t_{n+1})-\boldsymbol{z}^-(t_n)),\nabla_0\boldsymbol{e}^+_{n+1}\big) && (3.14)\\
&\quad\le C\Delta t^2(\nu_m-\nu)^2\gamma^{-1}\|\nabla\partial_t\boldsymbol{z}^-(\eta_n)\|^2+\gamma\|\nabla_0\boldsymbol{e}^+_{n+1}\|^2,\\
&\frac{\nu_m-\nu}{2}\big(\nabla(\boldsymbol{z}^+(t_{n+1})-\boldsymbol{z}^+(t_n)),\nabla_0\boldsymbol{e}^-_{n+1}\big) && (3.15)\\
&\quad\le C\Delta t^2(\nu_m-\nu)^2\gamma^{-1}\|\nabla\partial_t\boldsymbol{z}^+(\eta_n)\|^2+\gamma\|\nabla_0\boldsymbol{e}^-_{n+1}\|^2.
\end{aligned}
$$

Specifying $\gamma=\frac{\nu\nu_m}{8(\nu+\nu_m)}$, substituting (3.8)-(3.15) into (3.7), and rearranging gives

$$
\begin{aligned}
&\frac{1}{2\Delta t}(\|{}_0\boldsymbol{e}^+_{n+1}\|^2-\|{}_0\boldsymbol{e}^+_n\|^2)+\frac{1}{2\Delta t}(\|{}_0\boldsymbol{e}^-_{n+1}\|^2-\|{}_0\boldsymbol{e}^-_n\|^2) && (3.16)\\
&\quad+\frac{(\nu-\nu_m)^2}{4(\nu+\nu_m)}(\|\nabla_0\boldsymbol{e}^+_{n+1}\|^2-\|\nabla_0\boldsymbol{e}^+_n\|^2+\|\nabla_0\boldsymbol{e}^-_{n+1}\|^2-\|\nabla_0\boldsymbol{e}^-_n\|^2)\\
&\quad+\frac{\nu\nu_m}{2(\nu+\nu_m)}(\|\nabla_0\boldsymbol{e}^+_{n+1}\|^2+\|\nabla_0\boldsymbol{e}^-_{n+1}\|^2)\\
&\le C\frac{\nu+\nu_m}{\nu\nu_m}\Delta t^2\Big(\|\partial_t\boldsymbol{z}^+(\eta_n)\|^2\|\nabla\boldsymbol{z}^-(t_{n+1})\|^2_\infty+\|\partial_t\boldsymbol{z}^-(\eta_n)\|^2\|\nabla\boldsymbol{z}^+(t_n)\|^2_\infty\\
&\quad+\|\partial_{tt}\boldsymbol{z}^-(\eta_n)\|^2+\|\partial_{tt}\boldsymbol{z}^+(\eta_n)\|^2+(\nu_m-\nu)^2\|\nabla\partial_t\boldsymbol{z}^-(\eta_n)\|^2\\
&\quad+(\nu_m-\nu)^2\|\nabla\partial_t\boldsymbol{z}^+(\eta_n)\|^2\Big)\\
&\quad+C\frac{\nu+\nu_m}{\nu\nu_m}\big(\|{}_0\boldsymbol{e}^+_n\|^2+\|{}_0\boldsymbol{e}^+_n\|^2\big)\big(\|\nabla\boldsymbol{z}^+(t_{n+1})\|^2_\infty+\|\nabla\boldsymbol{z}^-(t_{n+1})\|^2_\infty\big).
\end{aligned}
$$

The proof is finished by multiplying (3.16) by $2\Delta t$, summing from $n=0$ to $N-1$, dropping nonnegative LHS terms, and applying Gronwall's inequality. □

3.2. Second order unconditionally stable SISDC partitioned scheme. Having shown the IMEX method is linear, unconditionally stable, modular and first order accurate in time we seek to develop a more accurate method that retains the 'good' qualities of the IMEX method. To this end we employ the spectral deferred correction technique (for further details of this technique see [13, 33, 21]).

The second order semi-implicit spectral deferred correction method is as follows: after computing first order approximations, $({}_0\boldsymbol{z}^\pm_n,{}_0p^\pm_n)$ and $({}_0\boldsymbol{z}^\pm_{n+1},{}_0p^\pm_{n+1})$ (using for example the IMEX method above) of (1.2) at time t_n and t_{n+1} respectively we seek to compute $({}_1\boldsymbol{z}^\pm_{n+1},{}_1p^\pm_{n+1})$ satisfying

$$
\begin{aligned}
&\frac{1}{\Delta t}({}_1\boldsymbol{z}^\pm_{n+1}-{}_1\boldsymbol{z}^\pm_n)-\frac{\nu+\nu_m}{2}\Delta_1\boldsymbol{z}^\pm_{n+1}-\frac{\nu-\nu_m}{2}\Delta_1\boldsymbol{z}^\pm_n\mp(\boldsymbol{B}_\circ\cdot\nabla)_1\boldsymbol{z}^\pm_{n+1}\\
&+({}_1\boldsymbol{z}^\mp_n\cdot\nabla)_1z^\pm_{n+1}+\nabla_1p^\pm_{n+1}=\mp(\boldsymbol{B}_\circ\cdot\nabla)_0z^\pm_{n+1}-\frac{\nu+\nu_m}{2}\Delta_0\boldsymbol{z}^\pm_{n+1}-\frac{\nu-\nu_m}{2}\Delta_0\boldsymbol{z}^\mp_n\\
&\pm\frac{1}{2}(\boldsymbol{B}_\circ\cdot\nabla)_0\boldsymbol{z}^\pm_{n+1}\pm\frac{1}{2}(\boldsymbol{B}_\circ\cdot\nabla)_0\boldsymbol{z}^\pm_n+\frac{\nu+\nu_m}{4}\Delta_0\boldsymbol{z}^\pm_{n+1}+\frac{\nu+\nu_m}{4}\Delta_0\boldsymbol{z}^\pm_n\\
&+\frac{\nu-\nu_m}{4}\Delta_0\boldsymbol{z}^\pm_{n+1}+\frac{\nu-\nu_m}{4}\Delta_0\boldsymbol{z}^\pm_n-\frac{1}{2}({}_0\boldsymbol{z}^\mp_{n+1}\cdot\nabla)_0\boldsymbol{z}^\pm_{n+1}-\frac{1}{2}({}_0\boldsymbol{z}^\mp_{n+1}\cdot\nabla)_0\boldsymbol{z}^\pm_n\\
&-\nabla_0p^\pm_{n+1}+\frac{1}{2}\nabla({}_0p^\pm_{n+1}+{}_0p^\pm_n)+\frac{1}{2}\boldsymbol{f}(t_{n+1})^\pm+\frac{1}{2}\boldsymbol{f}(t_n)^\pm,\\
&\nabla\cdot\boldsymbol{z}^\pm_{n+1}=0.
\end{aligned}
$$

LEMMA 3.2. *Given a final time $T > 0$ and timestep $\Delta t > 0$, let ${}_0\boldsymbol{z}_n^{\pm}$ denote the IMEX solution at time $t_n = n \times \Delta t$ for $n = 1, 2, ..., N$. Then the solutions to the SISDC method are unconditionally stable and satisfy*

$$
\begin{aligned}
&\|{}_1\boldsymbol{z}_N^{+}\|^2 + \|{}_1\boldsymbol{z}_N^{-}\|^2 + \Delta t \sum_{k=1}^{N} \frac{\nu\nu_m}{(\nu+\nu_m)}(\|\nabla {}_1\boldsymbol{z}_k^{+}\|^2 + \|\nabla {}_1\boldsymbol{z}_k^{-}\|^2) \\
&+ \Delta t \frac{(\nu-\nu_m)^2}{2(\nu+\nu_m)}\Big(\|\nabla {}_1\boldsymbol{z}_N^{+}\|^2 + \|\nabla {}_1\boldsymbol{z}_N^{-}\|^2\Big) \\
&\leq C\Delta t \frac{\nu+\nu_m}{\nu\nu_m} \sum_{k=0}^{N-1} \Big((\|\boldsymbol{B}_\circ\|_\infty + (\nu+\nu_m)^2 + (\nu-\nu_m)^2)(\|\nabla {}_0\boldsymbol{z}_{k+1}^{+}\|^2 + \|\nabla {}_0\boldsymbol{z}_k^{+}\|^2 \\
&+ \|\nabla {}_0\boldsymbol{z}_{k+1}^{-}\|^2 + \|\nabla {}_0\boldsymbol{z}_k^{-}\|^2) + (\|\nabla {}_0\boldsymbol{z}_{k+1}^{+}\|^2 + \|\nabla {}_0\boldsymbol{z}_k^{+}\|^2)(\|\nabla {}_0\boldsymbol{z}_{k+1}^{-}\|^2 + \|\nabla {}_0\boldsymbol{z}_k^{-}\|^2) \\
&+ \|\boldsymbol{f}^{+}(t_{k+1})\|^2 + \|\boldsymbol{f}^{+}(t_k)\|^2 + \|\boldsymbol{f}^{-}(t_{k+1})\|^2 + \|\boldsymbol{f}^{-}(t_k)\|^2\Big) \\
&+ \|\boldsymbol{z}(t_0)^{+}\|^2 + \|\boldsymbol{z}(t_0)^{-}\|^2 + \frac{\Delta t}{2}\frac{(\nu-\nu_m)^2}{\nu+\nu_m}(\|\nabla \boldsymbol{z}(t_0)^{+}\|^2 + \|\nabla \boldsymbol{z}(t_0)^{-}\|^2).
\end{aligned}
$$

Proof. Multiplying the SISDC momentum equations by ${}_1\boldsymbol{z}_{n+1}^{+}$ and ${}_1\boldsymbol{z}_{n+1}^{-}$ respectively, applying continuity equations and polarization identity, and adding the relations gives

$$
\begin{aligned}
&\frac{\|\boldsymbol{z}_{n+1}^{+}\|^2 - \|\boldsymbol{z}_n^{+}\|^2 + \|\boldsymbol{z}_{n+1}^{+} - \boldsymbol{z}_n^{+}\|^2}{2\Delta t} + \frac{\nu+\nu_m}{2}\|\nabla \boldsymbol{z}_{n+1}^{+}\|^2 + \frac{\nu-\nu_m}{2}(\nabla \boldsymbol{z}_n^{-}, \nabla \boldsymbol{z}_{n+1}^{+}) \qquad (3.17) \\
&+ \frac{\|\boldsymbol{z}_{n+1}^{-}\|^2 - \|\boldsymbol{z}_n^{-}\|^2 + \|\boldsymbol{z}_{n+1}^{-} - \boldsymbol{z}_n^{-}\|^2}{2\Delta t} + \frac{\nu+\nu_m}{2}\|\nabla \boldsymbol{z}_{n+1}^{-}\|^2 + \frac{\nu-\nu_m}{2}(\nabla \boldsymbol{z}_n^{+}, \nabla \boldsymbol{z}_{n+1}^{-}) \\
&= -((\boldsymbol{B}_\circ \cdot \nabla) {}_0\boldsymbol{z}_{n+1}^{+}, {}_1\boldsymbol{z}_{n+1}^{+}) + \frac{\nu+\nu_m}{4}(\nabla {}_0\boldsymbol{z}_{n+1}^{+}, \nabla {}_1\boldsymbol{z}_{n+1}^{+}) + \frac{\nu-\nu_m}{4}(\nabla {}_0\boldsymbol{z}_n^{-}, \nabla {}_1\boldsymbol{z}_{n+1}^{+}) \\
&+ \frac{1}{2}((\boldsymbol{B}_\circ \cdot \nabla) {}_0\boldsymbol{z}_{n+1}^{+}, {}_1\boldsymbol{z}_{n+1}^{+}) + \frac{1}{2}((\boldsymbol{B}_\circ \cdot \nabla) {}_0\boldsymbol{z}_n^{+}, {}_1\boldsymbol{z}_{n+1}^{+}) - \frac{\nu_m+\nu}{4}(\nabla {}_0\boldsymbol{z}_n^{-}, \nabla {}_1\boldsymbol{z}_{n+1}^{+}) \\
&- \frac{\nu-\nu_m}{4}(\nabla {}_0\boldsymbol{z}_{n+1}^{-}, \nabla {}_1\boldsymbol{z}_{n+1}^{+}) + (({}_0\boldsymbol{z}_n^{-} \cdot \nabla) {}_0\boldsymbol{z}_{n+1}^{+}), {}_1\boldsymbol{z}_{n+1}^{+}) \\
&- \frac{1}{2}(({}_0\boldsymbol{z}_{n+1}^{-} \cdot \nabla) {}_0\boldsymbol{z}_{n+1}^{+}, {}_1\boldsymbol{z}_{n+1}^{+}) - \frac{1}{2}(({}_0\boldsymbol{z}_n^{-} \cdot \nabla) {}_0\boldsymbol{z}_n^{+}, {}_1\boldsymbol{z}_{n+1}^{+}) + \frac{1}{2}(\boldsymbol{f}^{+}(t_{n+1}), {}_1\boldsymbol{z}_{n+1}^{+}) \\
&+ \frac{1}{2}(\boldsymbol{f}^{+}(t_n), {}_1\boldsymbol{z}_{n+1}^{+}) \\
&+ ((\boldsymbol{B}_\circ \cdot \nabla) {}_0\boldsymbol{z}_{n+1}^{-}, {}_1\boldsymbol{z}_{n+1}^{-}) + \frac{\nu+\nu_m}{4}(\nabla {}_0\boldsymbol{z}_{n+1}^{-}, \nabla {}_1\boldsymbol{z}_{n+1}^{-}) + \frac{\nu-\nu_m}{4}(\nabla {}_0\boldsymbol{z}_n^{+}, \nabla {}_1\boldsymbol{z}_{n+1}^{-}) \\
&- \frac{1}{2}((\boldsymbol{B}_\circ \cdot \nabla) {}_0\boldsymbol{z}_{n+1}^{-}, {}_1\boldsymbol{z}_{n+1}^{-}) - \frac{1}{2}((\boldsymbol{B}_\circ \cdot \nabla) {}_0\boldsymbol{z}_n^{-}, {}_1\boldsymbol{z}_{n+1}^{-}) - \frac{\nu_m+\nu}{4}(\nabla {}_0\boldsymbol{z}_n^{-}, \nabla {}_1\boldsymbol{z}_{n+1}^{-}) \\
&- \frac{\nu-\nu_m}{4}(\nabla {}_0\boldsymbol{z}_{n+1}^{-}, \nabla {}_1\boldsymbol{z}_{n+1}^{-}) + (({}_0\boldsymbol{z}_n^{+} \cdot \nabla) {}_0\boldsymbol{z}_{n+1}^{-}), {}_1\boldsymbol{z}_{n+1}^{-}) \\
&- \frac{1}{2}(({}_0\boldsymbol{z}_{n+1}^{+} \cdot \nabla) {}_0\boldsymbol{z}_{n+1}^{-}, {}_1\boldsymbol{z}_{n+1}^{-}) - \frac{1}{2}(({}_0\boldsymbol{z}_n^{+} \cdot \nabla) {}_0\boldsymbol{z}_n^{-}, {}_1\boldsymbol{z}_{n+1}^{-}) + \frac{1}{2}(\boldsymbol{f}^{-}(t_{n+1}), {}_1\boldsymbol{z}_{n+1}^{+}) \\
&+ \frac{1}{2}(\boldsymbol{f}^{-}(t_n), {}_1\boldsymbol{z}_{n+1}^{+}).
\end{aligned}
$$

Lower bounding dissipation terms on the LHS of (3.17) using the Cauchy-Schwarz inequality and polarization identity (for further details see [41]) and dropping non-

negative terms gives

$$
\begin{aligned}
&\frac{1}{2\Delta t}\big(\|{}_1\boldsymbol{z}_{n+1}^-\|^2-\|{}_1\boldsymbol{z}_n^+\|^2\big)+\frac{1}{2\Delta t}\big(\|{}_1\boldsymbol{z}_{n+1}^-\|^2-\|{}_1\boldsymbol{z}_n^-\|^2\big) \qquad (3.18)\\
&+\frac{\nu\nu_m}{2(\nu+\nu_m)}(\|\nabla_1\boldsymbol{z}_{n+1}^+\|^2+\|\nabla_1\boldsymbol{z}_{n+1}^-\|^2)+\frac{(\nu-\nu_m)^2}{4(\nu+\nu_m)}\Big(\|\nabla_1\boldsymbol{z}_{n+1}^+\|^2-\|\nabla_1\boldsymbol{z}_n^+\|^2\\
&+\|\nabla_1\boldsymbol{z}_{n+1}^-\|^2-\|\nabla_1\boldsymbol{z}_n^-\|^2\Big)\\
&\le-((\boldsymbol{B}_\circ\cdot\nabla)_0\boldsymbol{z}_{n+1}^+,{}_1\boldsymbol{z}_{n+1}^+)+\frac{\nu+\nu_m}{4}(\nabla_0\boldsymbol{z}_{n+1}^+,\nabla_1\boldsymbol{z}_{n+1}^+)+\frac{\nu-\nu_m}{4}(\nabla_0\boldsymbol{z}_n^-,\nabla_1\boldsymbol{z}_{n+1}^+)\\
&+\frac{1}{2}((\boldsymbol{B}_\circ\cdot\nabla)_0\boldsymbol{z}_{n+1}^+,{}_1\boldsymbol{z}_{n+1}^+)+\frac{1}{2}((\boldsymbol{B}_\circ\cdot\nabla)_0\boldsymbol{z}_n^+,{}_1\boldsymbol{z}_{n+1}^+)-\frac{\nu_m+\nu}{4}(\nabla_0\boldsymbol{z}_n^-,\nabla_1\boldsymbol{z}_{n+1}^+)\\
&-\frac{\nu-\nu_m}{4}(\nabla_0\boldsymbol{z}_{n+1}^-,\nabla_1\boldsymbol{z}_{n+1}^+)+((_0\boldsymbol{z}_n^-\cdot\nabla)_0\boldsymbol{z}_{n+1}^+),{}_1\boldsymbol{z}_{n+1}^+)\\
&-\frac{1}{2}((_0\boldsymbol{z}_{n+1}^-\cdot\nabla)_0\boldsymbol{z}_{n+1}^+,{}_1\boldsymbol{z}_{n+1}^+)-\frac{1}{2}((_0\boldsymbol{z}_n^-\cdot\nabla)_0\boldsymbol{z}_n^+,{}_1\boldsymbol{z}_{n+1}^+)+\frac{1}{2}(\boldsymbol{f}^+(t_{n+1}),{}_1\boldsymbol{z}_{n+1}^+)\\
&+\frac{1}{2}(\boldsymbol{f}^+(t_n),{}_1\boldsymbol{z}_{n+1}^+)\\
&+((\boldsymbol{B}_\circ\cdot\nabla)_0\boldsymbol{z}_{n+1}^-,{}_1\boldsymbol{z}_{n+1}^-)+\frac{\nu+\nu_m}{4}(\nabla_0\boldsymbol{z}_{n+1}^-,\nabla_1\boldsymbol{z}_{n+1}^-)+\frac{\nu-\nu_m}{4}(\nabla_0\boldsymbol{z}_n^+,|!\nabla_1\boldsymbol{z}_{n+1}^-)\\
&-\frac{1}{2}((\boldsymbol{B}_\circ\cdot\nabla)_0\boldsymbol{z}_{n+1}^-,{}_1\boldsymbol{z}_{n+1}^-)-\frac{1}{2}((\boldsymbol{B}_\circ\cdot\nabla)_0\boldsymbol{z}_n^-,{}_1\boldsymbol{z}_{n+1}^-)-\frac{\nu_m+\nu}{4}(\nabla_0\boldsymbol{z}_n^-,\nabla_1\boldsymbol{z}_{n+1}^-)\\
&-\frac{\nu-\nu_m}{4}(\nabla_0\boldsymbol{z}_{n+1}^-,\nabla_1\boldsymbol{z}_{n+1}^-)+((_0\boldsymbol{z}_n^+\cdot\nabla)_0\boldsymbol{z}_{n+1}^-),{}_1\boldsymbol{z}_{n+1}^-)\\
&-\frac{1}{2}((_0\boldsymbol{z}_{n+1}^+\cdot\nabla)_0\boldsymbol{z}_{n+1}^-,{}_1\boldsymbol{z}_{n+1}^-)-\frac{1}{2}((_0\boldsymbol{z}_n^+\cdot\nabla)_0\boldsymbol{z}_n^-,{}_1\boldsymbol{z}_{n+1}^-)+\frac{1}{2}(\boldsymbol{f}^-(t_{n+1}),{}_1\boldsymbol{z}_{n+1}^+)\\
&+\frac{1}{2}(\boldsymbol{f}^-(t_n),{}_1\boldsymbol{z}_{n+1}^+).
\end{aligned}
$$

Majorizing the RHS terms of (3.18) with standard inequalities gives

$$
\begin{aligned}
&\frac{1}{2\Delta t}\big(\|{}_1\boldsymbol{z}_{n+1}^-\|^2-\|{}_1\boldsymbol{z}_n^+\|^2\big)+\frac{1}{2\Delta t}\big(\|{}_1\boldsymbol{z}_{n+1}^-\|^2-\|{}_1\boldsymbol{z}_n^-\|^2\big)\\
&\quad+\frac{\nu\nu_m}{2(\nu+\nu_m)}(\|\nabla_1\boldsymbol{z}_{n+1}^+\|^2+\|\nabla_1\boldsymbol{z}_{n+1}^-\|^2)+\frac{(\nu-\nu_m)^2}{4(\nu+\nu_m)}\Big(\|\nabla_1\boldsymbol{z}_{n+1}^+\|^2-\|\nabla_1\boldsymbol{z}_n^+\|^2\\
&\quad+\|\nabla_1\boldsymbol{z}_{n+1}^-\|^2-\|\nabla_1\boldsymbol{z}_n^-\|^2\Big)\\
&\le C\frac{\nu+\nu_m}{(\nu-\nu_m)^2}\Big((\|\boldsymbol{B}_\circ\|_\infty+(\nu+\nu_m)^2+(\nu-\nu_m)^2)(\|\nabla_0\boldsymbol{z}_{n+1}^+\|^2+\|\nabla_0\boldsymbol{z}_n^+\|^2\\
&\quad+\|\nabla_0\boldsymbol{z}_{n+1}^-\|^2+\|\nabla_0\boldsymbol{z}_n^-\|^2)+(\|\nabla_0\boldsymbol{z}_{n+1}^+\|^2+\|\nabla_0\boldsymbol{z}_n^+\|^2)(\|\nabla_0\boldsymbol{z}_{n+1}^-\|^2+\|\nabla_0\boldsymbol{z}_n^-\|^2)\\
&\quad+\|\boldsymbol{f}_{n+1}^+\|^2+\|\boldsymbol{f}_n^+\|^2+\|\boldsymbol{f}_{n+1}^-\|^2+\|\boldsymbol{f}_n^-\|^2\Big).
\end{aligned}
$$

Multiplying by $2\Delta t$ and summing over timesteps yields the desired result. □

The purpose of adding the correction step is to develop a more accurate numerical method and so we seek to show that the SISDC solutions are in fact second order accurate. To this end we state and prove the following lemma, which is necessary for the proof of the accuracy of the SISDC method.

LEMMA 3.3. *Given a final time $T>0$ and timestep $\Delta t>0$ let ${}_0\boldsymbol{z}_n^\pm$ be the IMEX approximation to $\boldsymbol{z}(n\Delta t)^\pm$ for $n=1,2,...,N$. Let ${}_0\boldsymbol{e}_n^\pm=\boldsymbol{z}(t_n)^\pm-{}_0\boldsymbol{z}_n^\pm$ Provided the*

true solution satisfies the additional regularity assumptions

$$\partial_{ttt}\boldsymbol{z}^{\pm}\in L^2((0,T);L^2(\Omega)),\partial_t\boldsymbol{z}^{\pm}\in L^2((0,T);L^\infty(\Omega)),\nabla\partial_t\boldsymbol{z}^{\pm}\in L^2((0,T);L^\infty(\Omega))$$

the discrete time derivative of the IMEX is first order accurate in time and satisfies

$$\begin{aligned}
&\|\frac{{}_0\boldsymbol{e}_N^+-{}_0\boldsymbol{e}_{N-1}^+}{\Delta t}\|^2+\|\frac{{}_0\boldsymbol{e}_N^--{}_0\boldsymbol{e}_{N-1}^-}{\Delta t}\|^2\\
&\quad+\Delta t\frac{\nu\nu_m}{\nu+\nu_m}\sum_{n=0}^{N-1}\Big(\Big\|\frac{\nabla({}_0\boldsymbol{e}_N^+-{}_0\boldsymbol{e}_{N-1}^+)}{\Delta t}\Big\|^2+\Big\|\frac{\nabla({}_0\boldsymbol{e}_N^--{}_0\boldsymbol{e}_{N-1}^-)}{\Delta t}\Big\|^2\Big)\\
&\le C\frac{\nu+\nu_m}{\nu\nu_m}\exp\Big(\Delta t\sum_{n=0}^{N-1}(\|\nabla\boldsymbol{z}^+(t_n)\|_\infty^2+\|\nabla\boldsymbol{z}^-(t_{n+1})\|_\infty^2)\Big)\times\\
&\quad\times\Delta t\sum_{n=0}^{N-1}\Big(\Delta t^2\|\partial_{ttt}\boldsymbol{z}^+(\eta_n)\|^2+\Delta t^2\|\partial_{ttt}\boldsymbol{z}^-(\eta_n)\|^2+\Delta t^2\|\nabla\partial_t\boldsymbol{z}^+(\eta_n)\|_\infty^2\\
&\qquad+\Delta t^2\|\nabla\partial_t\boldsymbol{z}^-(\eta_n)\|^2\|\nabla\partial_t\boldsymbol{z}^+(\eta_n)\|^2+\Delta t^2\|\nabla\partial_{tt}\boldsymbol{z}^-(\eta_n)\|^2\|\nabla\boldsymbol{z}^+(t_n)\|^2\\
&\qquad+\Delta t^2\|\nabla\partial_{tt}\boldsymbol{z}^+(\eta_n)\|^2\|\nabla\partial_t\boldsymbol{z}^-(t_n)\|^2+\|\partial_t\boldsymbol{z}^-(\eta_n)\|_\infty^2\|\nabla{}_0\boldsymbol{e}_{n+1}^+\|^2\\
&\qquad+\|\nabla\partial_t\boldsymbol{z}^+(\eta_n)\|^2\|\nabla{}_0\boldsymbol{e}_n^-\|^2+\|\partial_t\boldsymbol{z}^+(\eta_n)\|_\infty^2\|\nabla{}_0\boldsymbol{e}_{n+1}^-\|^2\\
&\qquad+\|\nabla\partial_t\boldsymbol{z}^-(\eta_n)\|^2\|\nabla{}_0\boldsymbol{e}_n^+\|^2+\|\nabla{}_0\boldsymbol{e}_n^+\|^2+\|\nabla{}_0\boldsymbol{e}_n^-\|^2\Big).
\end{aligned}$$

Proof. For ease of notation we begin by defining $\boldsymbol{s}_{n+1}\pm:=\frac{1}{\Delta t}({}_0\boldsymbol{e}_{n+1}^{\pm}-{}_0\boldsymbol{e}_n^{\pm})$. Next consider (3.3) and (3.4) at timesteps $n+1$ and n. Subtracting (3.4) from (3.3) at timestep $n+1$ gives an equation involving ${}_0\boldsymbol{e}_{n+1}^+$, and subtracting (3.4) from (3.3) at timestep n gives an equation involving ${}_0\boldsymbol{e}_n^+$. We then subtract the new equations involving the IMEX errors to get an equation that involves $\boldsymbol{s}_{n+1}^+$. We similarly derive an equation involving $\boldsymbol{s}_{n+1}^-$. Multiplying the respective equations by $\frac{1}{\Delta t}\boldsymbol{s}_{n+1}^+$ and $\frac{1}{\Delta t}\boldsymbol{s}_{n+1}^-$, adding the equations, and reducing gives

$$\begin{aligned}
&\tfrac{1}{2\Delta t}(\|\boldsymbol{s}_{n+1}^+\|^2-\|\boldsymbol{s}_n^+\|^2)+\tfrac{1}{2\Delta t}(\|\boldsymbol{s}_{n+1}^-\|^2-\|\boldsymbol{s}_n^-\|^2)+\tfrac{\nu+\nu_m}{2}(\|\nabla\boldsymbol{s}_{n+1}^+\|^2+\|\nabla\boldsymbol{s}_{n+1}^-\|^2)\\
&\quad+\tfrac{(\nu-\nu_m)^2}{4(\nu+\nu_m)}\big(\|\nabla\boldsymbol{s}_{n+1}^+\|^2-\|\nabla\boldsymbol{s}_n^+\|^2+\|\nabla\boldsymbol{s}_{n+1}^-\|^2-\|\nabla\boldsymbol{s}_n^-\|^2\big) \qquad (3.19)\\
&=\Big(\frac{1}{\Delta t}(({}_0\boldsymbol{z}_n^-\cdot\nabla){}_0\boldsymbol{z}_{n+1}^+,\boldsymbol{s}_{n+1}^+)-\frac{1}{\Delta t}((\boldsymbol{z}^-(t_n)\cdot\nabla)\boldsymbol{z}^+(t_{n+1}),\boldsymbol{s}_{n+1}^+)\\
&\quad+\frac{1}{\Delta t}((\boldsymbol{z}^-(t_{n-1})\cdot\nabla)\boldsymbol{z}^+(t_n),\boldsymbol{s}_{n+1}^+)-\frac{1}{\Delta t}(({}_0\boldsymbol{z}_{n-1}^-\cdot\nabla){}_0\boldsymbol{z}_n^+,\boldsymbol{s}_{n+1}^+)\Big)\\
&\quad+\Big(\frac{1}{\Delta t}(({}_0\boldsymbol{z}_n^+\cdot\nabla){}_0\boldsymbol{z}_{n+1}^-,\boldsymbol{s}_{n+1}^-)-\frac{1}{\Delta t}((\boldsymbol{z}^+(t_n)\cdot\nabla)\boldsymbol{z}^-(t_{n+1}),\boldsymbol{s}_{n+1}^-)\\
&\quad+\frac{1}{\Delta t}((\boldsymbol{z}^+(t_{n-1})\cdot\nabla)\boldsymbol{z}^-(t_n),\boldsymbol{s}_{n+1}^-)-\frac{1}{\Delta t}(({}_0\boldsymbol{z}_{n-1}^+\cdot\nabla){}_0\boldsymbol{z}_n^-,\boldsymbol{s}_{n+1}^-)\Big)\\
&\quad+\Big(\frac{1}{\Delta t}((\boldsymbol{z}^-(t_n)\cdot\nabla)\boldsymbol{z}^+(t_{n+1}),\boldsymbol{s}_{n+1}^+)-\frac{1}{\Delta t}((\boldsymbol{z}^-(t_{n+1})\cdot\nabla)\boldsymbol{z}^+(t_{n+1}),\boldsymbol{s}_{n+1}^+)\\
&\quad+\frac{1}{\Delta t}((\boldsymbol{z}^-(t_n)\cdot\nabla)\boldsymbol{z}^+(t_n),\boldsymbol{s}_{n+1}^+)-\frac{1}{\Delta t}((\boldsymbol{z}^-(t_{n-1})\cdot\nabla)\boldsymbol{z}^+(t_n),\boldsymbol{s}_{n+1}^+)\Big)\\
&\quad+\Big(\frac{1}{\Delta t}((\boldsymbol{z}^+(t_n)\cdot\nabla)\boldsymbol{z}^-(t_{n+1}),\boldsymbol{s}_{n+1}^-)-\frac{1}{\Delta t}((\boldsymbol{z}^+(t_{n+1})\cdot\nabla)\boldsymbol{z}^-(t_{n+1}),\boldsymbol{s}_{n+1}^-)\\
&\quad+\frac{1}{\Delta t}((\boldsymbol{z}^+(t_n)\cdot\nabla)\boldsymbol{z}^-(t_n),\boldsymbol{s}_{n+1}^-)-\frac{1}{\Delta t}((\boldsymbol{z}^+(t_{n-1})\cdot\nabla)\boldsymbol{z}^-(t_n),\boldsymbol{s}_{n+1}^-)\Big)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Delta t}\Big(\frac{\boldsymbol{z}^+(t_{n+1})-\boldsymbol{z}^+(t_n)}{\Delta t}-\partial_t\boldsymbol{z}^+(t_{n+1}),\boldsymbol{s}^+_{n+1}\Big)\\
&\quad-\frac{1}{\Delta t}\Big(\frac{\boldsymbol{z}^+(t_n)-\boldsymbol{z}^+(t_{n-1})}{\Delta t}-\partial_t\boldsymbol{z}^+(t_n),\boldsymbol{s}^+_{n+1}\Big)\\
&+\frac{1}{\Delta t}\Big(\frac{\boldsymbol{z}^-(t_{n+1})-\boldsymbol{z}^-(t_n)}{\Delta t}-\partial_t\boldsymbol{z}^-(t_{n+1}),\boldsymbol{s}^-_{n+1}\Big)\\
&\quad-\frac{1}{\Delta t}\Big(\frac{\boldsymbol{z}^-(t_n)-\boldsymbol{z}^-(t_{n-1})}{\Delta t}-\partial_t\boldsymbol{z}^-(t_n),\boldsymbol{s}^-_{n+1}\Big).
\end{aligned}
$$

After bounding the nonlinear terms of (3.19) (see A.1) applying Taylor series to the remaining linear terms of (3.19) yields

$$
\begin{aligned}
&\frac{1}{\Delta t}\Big|\Big(\frac{\boldsymbol{z}^+(t_{n+1})-\boldsymbol{z}^+(t_n)}{\Delta t}-\partial_t\boldsymbol{z}^+(t_{n+1}),\boldsymbol{s}^+_{n+1}\Big)\\
&\qquad-\Big(\frac{\boldsymbol{z}^+(t_n)-\boldsymbol{z}^+(t_{n-1})}{\Delta t}-\partial_t\boldsymbol{z}^+(t_n),\boldsymbol{s}^+_{n+1}\Big)\Big|\\
&\le C\gamma^{-1}\Delta t^2\|\partial_{ttt}\boldsymbol{z}^+(\eta_n)\|^2+\gamma\|\nabla\boldsymbol{s}^+_{n+1}\|^2,
\end{aligned}
\tag{3.20}
$$

$$
\begin{aligned}
&\frac{1}{\Delta t}\Big|\Big(\frac{\boldsymbol{z}^-(t_{n+1})-\boldsymbol{z}^-(t_n)}{\Delta t}-\partial_t\boldsymbol{z}^-(t_{n+1}),\boldsymbol{s}^-_{n+1}\Big)\\
&\qquad-\Big(\frac{\boldsymbol{z}^-(t_n)-\boldsymbol{z}^-(t_{n-1})}{\Delta t}-\partial_t\boldsymbol{z}^-(t_n),\boldsymbol{s}^-_{n+1}\Big)\Big|\\
&\le C\gamma^{-1}\Delta t^2\|\partial_{ttt}\boldsymbol{z}^-(\eta_n)\|^2+\gamma\|\nabla\boldsymbol{s}^-_{n+1}\|^2.
\end{aligned}
\tag{3.21}
$$

Having bounded all RHS terms of (3.19) we continue by choosing $\gamma=\frac{\nu\nu_m}{14(\nu+\nu_m)}$, substituting (A.7), (A.8), (A.9), (A.10), (3.20), and (3.21) into (3.19), and rearranging. Multiplying the resulting equation by $2\Delta t$ gives

$$
\begin{aligned}
&(\|\boldsymbol{s}^+_{n+1}\|^2-\|\boldsymbol{s}^+_n\|^2)+(\|\boldsymbol{s}^-_{n+1}\|^2-\|\boldsymbol{s}^-_n\|^2)+\Delta t\frac{\nu\nu_m}{(\nu+\nu_m)}(\|\nabla\boldsymbol{s}^+_{n+1}\|^2+\|\nabla\boldsymbol{s}^-_{n+1}\|^2)\\
&\quad+\Delta t\frac{\nu\nu_m}{2(\nu+\nu_m)}(\|\nabla\boldsymbol{s}^+_{n+1}\|^2-\|\nabla\boldsymbol{s}^+_n\|^2+\|\nabla\boldsymbol{s}^-_{n+1}\|^2-\|\nabla\boldsymbol{s}^-_n\|^2)\\
&\le C\Delta t\frac{\nu+\nu_m}{\nu\nu_m}\Big(\|\boldsymbol{s}^+_n\|^2+\|\boldsymbol{s}^-_n\|^2\Big)\Big(\|\nabla\boldsymbol{z}^+(t_{n+1})\|^2_\infty+\|\nabla\boldsymbol{z}^-(t_{n+1})\|^2_\infty\Big)\\
&\quad+C\Delta t\frac{\nu+\nu_m}{\nu\nu_m}\Big(\|\partial_t\boldsymbol{z}^-(\eta_n)\|^2_\infty\|\nabla{}_0\boldsymbol{e}^+_{n+1}\|^2+\|{}_0\boldsymbol{e}^-_{n-1}\|^2\|\nabla\partial_t\boldsymbol{z}^+(\eta_n)\|^2_\infty\\
&\qquad+\|\partial_t\boldsymbol{z}^+(\eta_n)\|^2_\infty\|\nabla{}_0\boldsymbol{e}^-_{n+1}\|^2+\|{}_0\boldsymbol{e}^+_{n-1}\|^2\|\nabla\partial_t\boldsymbol{z}^-(\eta_n)\|^2_\infty\\
&\qquad+\Delta t^2\|\nabla\partial_t\boldsymbol{z}^-(\eta_n)\|^2\|\nabla\partial_t\boldsymbol{z}^+(\eta_n)\|^2+\Delta t^2\|\nabla\partial_{tt}\boldsymbol{z}^-(\eta_n)\|^2\|\nabla\boldsymbol{z}^+(t_n)\|^2\\
&\qquad+\Delta t^2\|\nabla\partial_t\boldsymbol{z}^+(\eta_n)\|^2\|\nabla\partial_t\boldsymbol{z}^-(\eta_n)\|^2+\Delta t^2\|\nabla\partial_{tt}\boldsymbol{z}^+(\eta_n)\|^2\|\nabla\boldsymbol{z}^-(t_n)\|^2\\
&\qquad+\Delta t^2\|\partial_{ttt}\boldsymbol{z}^+(\eta_n)\|^2+\Delta t^2\|\partial_{ttt}\boldsymbol{z}^-(\eta_n)\|^2\Big)\\
&\quad+C\frac{\nu+\nu_m}{\nu\nu_m}\Delta t\|\nabla s^+_n\|^2\|\nabla{}_0\boldsymbol{e}^-_{n+1}\|^4+C\frac{\nu+\nu_m}{\nu\nu_m}\Delta t\|\nabla s^-_n\|^2\|\nabla{}_0\boldsymbol{e}^+_{n+1}\|^4.
\end{aligned}
\tag{3.22}
$$

To finish deriving the error bound requires an application of Gronwall's inequality. However, there are two subtleties that prevent us from doing this immediately. The first complication is the last two terms of (3.22) involve $\nabla\boldsymbol{s}^\pm_n$, and so they require further treatment. Recall that ${}_0\boldsymbol{e}^\pm_n$ denotes the error in the IMEX solution, and so we have the following

$$
\Delta t\|\nabla{}_0\boldsymbol{e}^+_{n+1}\|^4=\frac{(\Delta t\|\nabla{}_0\boldsymbol{e}^\pm_{n+1}\|^2)^2}{\Delta t}\le C\Delta t^3.
\tag{3.23}
$$

Using (3.23) we derive bounds for the last two terms of (3.22) as follows

$$\Delta t\|\nabla s_n^{\pm}\|^2\|\nabla_0 \boldsymbol{e}_{n+1}^{\mp}\|^4 \le C\Delta t^3\|\nabla \boldsymbol{s}_n^{\pm}\|^2 \le C\Delta t\|\nabla \boldsymbol{e}_n^{\pm}\|^2 + C\Delta t\|\nabla \boldsymbol{e}_{n-1}^{\pm}\|^2. \tag{3.24}$$

The other subtlety that requires our attention is it is only possible to sum from (3.22) $n = 1$ to $N-1$, because $\boldsymbol{s}_0^{\pm} = \frac{1}{\Delta t}({}_0\boldsymbol{e}_0{}^{\pm} - {}_0\boldsymbol{e}_{-1}{}^{\pm})$ is not defined, and this leaves nonpositive terms on the LHS that must be dealt with. After we sum (3.22) from $n = 1$ to $N-1$ we are left with the terms $-\|\boldsymbol{s}_1^+\|^2$, $-\|\nabla \boldsymbol{s}_1^+\|^2$, $-\|\boldsymbol{s}_1^-\|^2$, and $-\|\nabla \boldsymbol{s}_1^-\|^2$ on the LHS. To apply Gronwall's inequality requires moving these terms to the RHS, and to yield the desired result we need these bounded by a multiple of Δt^2. Recall that the IMEX method is a first order accurate method, which implies the local error $({}_0\boldsymbol{e}_1^{\pm})$ in the method is second order accurate. Thus, we have the following bound

$$\|\boldsymbol{s}_1^+\|^2 = \|\frac{1}{\Delta t}({}_0\boldsymbol{e}_1^+ - {}_0\boldsymbol{e}_0^+)\|^2 = \|\frac{1}{\Delta t}{}_0\boldsymbol{e}_1^+\|^2 \le C\Delta t^2. \tag{3.25}$$

Bounds for $\|\nabla \boldsymbol{s}_1^+\|^2$, $\|\boldsymbol{s}_1^-\|^2$, and $\|\nabla \boldsymbol{s}_1^-\|^2$ can be derived similarly.

Substituting (3.24) in to (3.22), summing from $n = 1$ to $N-1$, rearranging and applying the discrete Gronwall lemma finishes the proof. ∎

We now state and prove the main result, that solutions found with the SISDC method are second order accurate.

THEOREM 3.4. *Given final time $T > 0$ and timestep $\Delta t > 0$ let ${}_0\boldsymbol{z}_n^{\pm}$ and ${}_1\boldsymbol{z}_n^{\pm}$ respectively denote the IMEX and SISDC approximations to $\boldsymbol{z}^{\pm}$ at time $t_n = n\Delta t$ for $n = 1, 2, ..., N$, and let ${}_1\boldsymbol{e}_n^{\pm} := \boldsymbol{z}^{\pm}(t_n) - {}_1\boldsymbol{z}_n^{\pm}$. Additionally, we let $\boldsymbol{s}_{n+1}^{\pm} = \frac{1}{\Delta t}({}_0\boldsymbol{e}_{n+1}^{\pm} - {}_0\boldsymbol{e}_n^{\pm})$ for $n = 1, 2, ..., N-1$. Provided the true solution satisfies the additional regularity*

$$\nabla \boldsymbol{z}^{\pm} \in L^2((0,T); L^2), \partial_{tt}(\boldsymbol{z}^{\mp}\cdot\nabla)\boldsymbol{z}^{\pm} \in L^2((0,T); L^2),$$
$$\partial_{tt}\nabla \boldsymbol{z}^{\pm} \in L^2((0,T); L^2), \text{ and } \partial_{tt}\boldsymbol{f}^+, \partial_{tt}\boldsymbol{f}^- \in L^2((0,T); L^2),$$

then the SISDC approximation is second order accurate in time and satisfies

$$\begin{aligned}
&\|{}_1\boldsymbol{e}_N^+\|^2 + \|{}_1\boldsymbol{e}_N^-\|^2 + \Delta t\frac{\nu\nu_m}{\nu+\nu_m}\sum_{n=0}^{N}(\|\nabla {}_1\boldsymbol{e}_n^+\|^2 + \|\nabla {}_1\boldsymbol{e}_n^-\|^2)\\
&\le C\frac{\nu+\nu_m}{\nu\nu_m}\exp\Big(\Delta t\sum_{n=0}^{N-1}\|\nabla \boldsymbol{z}^+(t_n)\|_\infty^2 + \|\nabla \boldsymbol{z}^-(t_n)\|_\infty^2\Big)\times\\
&\times\Big(\Delta t\sum_{n=0}^{N-1} C\frac{\nu+\nu_m}{\nu\nu_m}\{\Delta t^2\|\nabla \boldsymbol{B}_\circ\|^2\|\nabla \boldsymbol{s}_{n+1}^+\|^2 + \Delta t^2\|\nabla \boldsymbol{B}_\circ\|^2\|\nabla \boldsymbol{s}_{n+1}^-\|^2\\
&\quad+\Delta t^2\|\nabla \boldsymbol{z}^-(t_n)\|^2\|\nabla \boldsymbol{s}_{n+1}^+\|^2+\Delta t^2\|\nabla_0 \boldsymbol{e}_n^-\|^2\|\nabla \boldsymbol{s}_{n+1}^+\|^2+\Delta t^2\|\nabla_0 \boldsymbol{z}_n^-\|^2\|\nabla \boldsymbol{s}_{n+1}^+\|^2\\
&\quad+\Delta t^2\|\nabla \boldsymbol{s}_{n+1}^-\|^2\|\nabla_0 \boldsymbol{e}_{n+1}^+\|^2+\Delta t^2\|\nabla\partial_t \boldsymbol{z}^-(\eta_n)\|^2\|\nabla_0 \boldsymbol{e}_{n+1}^+\|^2\\
&\quad+\Delta t^2\|\nabla \boldsymbol{s}_{n+1}^-\|^2\|\nabla \boldsymbol{z}^+(t_{n+1})\|^2+\Delta t^2\|\nabla_0 \boldsymbol{e}_{n+1}^-\|^2\|\nabla\partial_t \boldsymbol{z}^+(\eta_n)\|^2\\
&\quad+\Delta t^2\|\nabla \boldsymbol{s}_{n+1}^-\|^2\|\nabla \boldsymbol{z}^+(t_n)\|^2+\Delta t^2\|\nabla \boldsymbol{z}^+(t_n)\|^2\|\nabla \boldsymbol{s}_{n+1}^-\|^2\\
&\quad+\Delta t^2\|\nabla_0 \boldsymbol{e}_n^+\|^2\|\nabla \boldsymbol{s}_{n+1}^-\|^2+\Delta t^2\|\nabla_0 \boldsymbol{z}_n^+\|^2\|\nabla \boldsymbol{s}_{n+1}^-\|^2+\Delta t^2\|\nabla \boldsymbol{s}_{n+1}^+\|^2\|\nabla_0 \boldsymbol{e}_{n+1}^-\|^2\\
&\quad+\Delta t^2\|\nabla\partial_t \boldsymbol{z}^+(\eta_n)\|^2\|\nabla_0 \boldsymbol{e}_{n+1}^-\|^2 + \Delta t^2\|\nabla \boldsymbol{s}_{n+1}^+\|^2\|\nabla \boldsymbol{z}^-(t_{n+1})\|^2\\
&\quad+\Delta t^2\|\nabla_0 \boldsymbol{e}_{n+1}^+\|^2\|\nabla\partial_t \boldsymbol{z}^-(\eta_n)\|^2 + \Delta t^2\|\nabla \boldsymbol{s}_{n+1}^+\|^2\|\nabla \boldsymbol{z}^-(t_n)\|^2\\
&\quad+\Delta t^2(\nu+\nu_m)^2\|\nabla \boldsymbol{s}_{n+1}^+\|^2+\Delta t^2(\nu-\nu_m)^2\|\nabla \boldsymbol{s}_{n+1}^-\|^2+\Delta t^2(\nu+\nu_m)^2\|\nabla \boldsymbol{s}_{n+1}^-\|^2\\
&\quad+\Delta t^2(\nu-\nu_m)^2\|\nabla \boldsymbol{s}_{n+1}^+\|^2+\Delta t^4\|\partial_{tt}\boldsymbol{F}(\eta_n)\|^2+\Delta t^4\|\partial_{tt}\boldsymbol{F}(\eta_n)\|^2\}\Big).
\end{aligned}$$

Proof. We begin the proof by defining $\boldsymbol{F}^+$ and $\boldsymbol{F}^-$ as follows

$$\boldsymbol{F}^+ := (\boldsymbol{B}_\circ \cdot \nabla)\boldsymbol{z}^+ - (\boldsymbol{z}^- \cdot \nabla)\boldsymbol{z}^+ + \frac{\nu+\nu_m}{2}\Delta \boldsymbol{z}^+ + \frac{\nu-\nu_m}{2}\Delta \boldsymbol{z}^- + \boldsymbol{f}^+ - \nabla p, \tag{3.26}$$

$$\boldsymbol{F}^- := -(\boldsymbol{B}_\circ \cdot \nabla)\boldsymbol{z}^+ - (\boldsymbol{z}^- \cdot \nabla)\boldsymbol{z}^+ + \frac{\nu+\nu_m}{2}\Delta \boldsymbol{z}^+ + \frac{\nu-\nu_m}{2}\Delta \boldsymbol{z}^- + \boldsymbol{f}^- - \nabla p. \tag{3.27}$$

We may express the first continuous momentum equation as

$$\frac{1}{\Delta t}(\boldsymbol{z}^+(t_{n+1}) - \boldsymbol{z}^+(t_n)) = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} \boldsymbol{F}^+ d\tau.$$

Applying the trapezoid rule to the integral gives

$$\begin{aligned}
&\frac{1}{\Delta t}(\boldsymbol{z}^+(t_{n+1}) - \boldsymbol{z}^+(t_n)) - \frac{\nu+\nu_m}{2}\Delta \boldsymbol{z}^+(t_{n+1}) - \frac{\nu-\nu_m}{2}\Delta \boldsymbol{z}^-(t_n) \\
&= \frac{1}{2}(\boldsymbol{B}_\circ \cdot \nabla)\boldsymbol{z}^+(t_{n+1}) + \frac{1}{2}(\boldsymbol{B}_\circ \cdot \nabla)\boldsymbol{z}^+(t_n) - \frac{1}{2}(\boldsymbol{z}^-(t_{n+1}) \cdot \nabla)\boldsymbol{z}^+(t_{n+1}) \\
&\quad - \frac{1}{2}(\boldsymbol{z}^-(t_n) \cdot \nabla)\boldsymbol{z}^+(t_n) - \frac{\nu+\nu_m}{4}\Delta \boldsymbol{z}^+(t_{n+1}) + \frac{\nu+\nu_m}{4}\Delta \boldsymbol{z}^+(t_n) \\
&\quad + \frac{\nu-\nu_m}{4}\Delta \boldsymbol{z}^-(t_{n+1}) - \frac{\nu-\nu_m}{4}\Delta \boldsymbol{z}^-(t_n) + \frac{1}{2}\boldsymbol{f}^+(t_{n+1}) + \frac{1}{2}\boldsymbol{f}^+(t_n) \\
&\quad - \frac{1}{2}\nabla p(t_{n+1}) - \frac{1}{2}\nabla p(t_n) + C\Delta t^2 \partial_{tt}\boldsymbol{F}^+(\eta_n).
\end{aligned} \tag{3.28}$$

The first momentum equation for the SISDC method is

$$\begin{aligned}
&\frac{1}{\Delta t}({}_1\boldsymbol{z}^+_{n+1} - {}_1\boldsymbol{z}^+_n) - \frac{\nu+\nu_m}{2}\Delta_1 \boldsymbol{z}^+_{n+1} - \frac{\nu-\nu_m}{2}\Delta_1 \boldsymbol{z}^+_n \\
&= (\boldsymbol{B}_\circ \cdot \nabla)_1 \boldsymbol{z}^+_{n+1} - (\boldsymbol{B}_\circ \cdot \nabla)_0 \boldsymbol{z}^+_{n+1} - \frac{\nu+\nu_m}{2}\Delta_0 \boldsymbol{z}^+_{n+1} - \frac{\nu-\nu_m}{2}\Delta_0 \boldsymbol{z}^-_n \\
&\quad + \frac{1}{2}(\boldsymbol{B}_\circ \cdot \nabla)_0 \boldsymbol{z}^+_{n+1} + \frac{1}{2}(\boldsymbol{B}_\circ \cdot \nabla)_0 \boldsymbol{z}^+_n + \frac{\nu+\nu_m}{4}\Delta_0 \boldsymbol{z}^+_{n+1} + \frac{\nu+\nu_m}{4}\Delta_0 \boldsymbol{z}^+_n \\
&\quad + \frac{\nu-\nu_m}{4}\Delta_0 \boldsymbol{z}^-_{n+1} + \frac{\nu-\nu_m}{4}\Delta_0 \boldsymbol{z}^-_n - ({}_1\boldsymbol{z}^-_n \cdot \nabla)_1 \boldsymbol{z}^+_{n+1} + ({}_0\boldsymbol{z}^-_n \cdot \nabla)_0 \boldsymbol{z}^+_{n+1} \\
&\quad - \frac{1}{2}({}_0\boldsymbol{z}^-_{n+1} \cdot \nabla)_0 \boldsymbol{z}^+_{n+1} - \frac{1}{2}({}_0\boldsymbol{z}^-_n \cdot \nabla)_0 \boldsymbol{z}^+_n + \frac{1}{2}\boldsymbol{f}^+(t_{n+1}) + \frac{1}{2}\boldsymbol{f}^+(t_n).
\end{aligned} \tag{3.29}$$

We may similarly express the second momentum equation as

$$\frac{1}{\Delta t}(\boldsymbol{z}^-(t_{n+1}) - \boldsymbol{z}^-(t_n)) = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} \boldsymbol{F}^- d\tau.$$

Applying the trapezoid rule to the integral gives

$$\begin{aligned}
&\frac{1}{\Delta t}(\boldsymbol{z}^-(t_{n+1}) - \boldsymbol{z}^-(t_n)) - \frac{\nu+\nu_m}{2}\Delta \boldsymbol{z}^-(t_{n+1}) - \frac{\nu-\nu_m}{2}\Delta \boldsymbol{z}^+(t_n) \\
&= -\frac{1}{2}(\boldsymbol{B}_\circ \cdot \nabla)\boldsymbol{z}^+(t_{n+1}) - \frac{1}{2}(\boldsymbol{B}_\circ \cdot \nabla)\boldsymbol{z}^+(t_n) - \frac{1}{2}(\boldsymbol{z}^+(t_{n+1}) \cdot \nabla)\boldsymbol{z}^-(t_{n+1}) \\
&\quad - \frac{1}{2}(\boldsymbol{z}^+(t_n) \cdot \nabla)\boldsymbol{z}^-(t_n) - \frac{\nu+\nu_m}{4}\Delta \boldsymbol{z}^-(t_{n+1}) + \frac{\nu+\nu_m}{4}\Delta \boldsymbol{z}^-(t_n) \\
&\quad + \frac{\nu-\nu_m}{4}\Delta \boldsymbol{z}^+(t_{n+1}) - \frac{\nu-\nu_m}{4}\Delta \boldsymbol{z}^+(t_n) + \frac{1}{2}\boldsymbol{f}^-(t_{n+1})
\end{aligned} \tag{3.30}$$

$$+\frac{1}{2}\boldsymbol{f}^-(t_n)-\frac{1}{2}\nabla p(t_{n+1})-\frac{1}{2}\nabla p(t_n)+C\Delta t^2\partial_{tt}\boldsymbol{F}^-(\eta_n).$$

The second momentum equation for the SISDC method is

$$\begin{aligned}
&\frac{1}{\Delta t}({}_1\boldsymbol{z}^-_{n+1}-{}_1\boldsymbol{z}^-_n)-\frac{\nu+\nu_m}{2}\Delta_1\boldsymbol{z}^-_{n+1}-\frac{\nu-\nu_m}{2}\Delta_1\boldsymbol{z}^-_n \qquad (3.31)\\
&=-(\boldsymbol{B}_\circ\cdot\nabla){}_1\boldsymbol{z}^+_{n+1}+(\boldsymbol{B}_\circ\cdot\nabla){}_0\boldsymbol{z}^+_{n+1}-\frac{\nu+\nu_m}{2}\Delta_0\boldsymbol{z}^+_{n+1}-\frac{\nu-\nu_m}{2}\Delta_0\boldsymbol{z}^-_n\\
&\quad-\frac{1}{2}(\boldsymbol{B}_\circ\cdot\nabla){}_0\boldsymbol{z}^+_{n+1}-\frac{1}{2}(\boldsymbol{B}_\circ\cdot\nabla){}_0\boldsymbol{z}^+_n+\frac{\nu+\nu_m}{4}\Delta_0\boldsymbol{z}^+_{n+1}+\frac{\nu+\nu_m}{4}\Delta_0\boldsymbol{z}^+_n\\
&\quad+\frac{\nu-\nu_m}{4}\Delta_0\boldsymbol{z}^-_{n+1}+\frac{\nu-\nu_m}{4}\Delta_0\boldsymbol{z}^-_n-({}_1\boldsymbol{z}^+_n\cdot\nabla){}_1\boldsymbol{z}^-_{n+1}+({}_0\boldsymbol{z}^+_n\cdot\nabla){}_0\boldsymbol{z}^-_{n+1}\\
&\quad-\frac{1}{2}({}_0\boldsymbol{z}^+_{n+1}\cdot\nabla){}_0\boldsymbol{z}^-_{n+1}-\frac{1}{2}({}_0\boldsymbol{z}^+_n\cdot\nabla){}_0\boldsymbol{z}^-_n+\frac{1}{2}\boldsymbol{f}^-(t_{n+1})+\frac{1}{2}\boldsymbol{f}^-(t_n).
\end{aligned}$$

Subtracting (3.29) from (3.28) and multiplying the result by ${}_1\boldsymbol{e}^+_{n+1}$ gives an equation in terms of ${}_1\boldsymbol{e}^+_{n+1}$. Similarly, subtracting (3.31) from (3.30) and multiplying the result by ${}_1\boldsymbol{e}^-_{n+1}$ gives an equation in terms of ${}_1\boldsymbol{e}^-_{n+1}$. Adding these relations, reducing, and lower bounding dissipation terms gives

$$\begin{aligned}
&\frac{1}{2\Delta t}(\|{}_1\boldsymbol{e}^+_{n+1}\|^2-\|{}_1\boldsymbol{e}^+_n\|^2)+\frac{1}{2\Delta t}(\|{}_1\boldsymbol{e}^-_{n+1}\|^2-\|{}_1\boldsymbol{e}^-_n\|^2) \qquad (3.32)\\
&\quad+\frac{\nu\nu_m}{\nu+\nu_m}(\|\nabla{}_1\boldsymbol{e}^+_{n+1}\|^2+\|\nabla{}_1\boldsymbol{e}^-_{n+1}\|^2)\\
&\quad+\frac{(\nu-\nu_m)^2}{4(\nu+\nu_m)}(\|\nabla{}_1\boldsymbol{e}^+_{n+1}\|^2-\|\nabla{}_1\boldsymbol{e}^+_n\|^2+\|\nabla{}_1\boldsymbol{e}^-_{n+1}\|^2-\|\nabla{}_1\boldsymbol{e}^-_n\|^2)\\
&\le\Big|\Big(\frac{1}{2}((\boldsymbol{B}_\circ\cdot\nabla)\boldsymbol{z}^+(t_{n+1}),{}_1\boldsymbol{e}^+_{n+1})+\frac{1}{2}((\boldsymbol{B}_\circ\cdot\nabla)\boldsymbol{z}^+(t_n),{}_1\boldsymbol{e}^+_{n+1})\\
&\quad-((\boldsymbol{B}_\circ\cdot\nabla){}_1\boldsymbol{z}^+_{n+1},{}_1\boldsymbol{e}^+_{n+1})+\frac{1}{2}((\boldsymbol{B}_\circ\cdot\nabla){}_0\boldsymbol{z}^+_{n+1},{}_1\boldsymbol{e}^+_{n+1})-\frac{1}{2}((\boldsymbol{B}_\circ\cdot\nabla){}_0\boldsymbol{z}^+_n,{}_1\boldsymbol{e}^+_{n+1})\Big)\\
&\quad+\Big(-\frac{1}{2}((\boldsymbol{B}_\circ\cdot\nabla)\boldsymbol{z}^-(t_{n+1}),{}_1\boldsymbol{e}^-_{n+1})-\frac{1}{2}((\boldsymbol{B}_\circ\cdot\nabla)\boldsymbol{z}^-(t_n),{}_1\boldsymbol{e}^-_{n+1})\\
&\quad+((\boldsymbol{B}_\circ\cdot\nabla){}_1\boldsymbol{z}^-_{n+1},{}_1\boldsymbol{e}^-_{n+1})-\frac{1}{2}((\boldsymbol{B}_\circ\cdot\nabla){}_0\boldsymbol{z}^-_{n+1},{}_1\boldsymbol{e}^-_{n+1})\\
&\quad+\Big(-\frac{1}{2}((\boldsymbol{z}^-(t_{n+1})\cdot\nabla)\boldsymbol{z}^+(t_{n+1}),{}_1\boldsymbol{e}^+_{n+1})-\frac{1}{2}((\boldsymbol{z}^-(t_n)\cdot\nabla)\boldsymbol{z}^+(t_n),{}_1\boldsymbol{e}^+_{n+1})\\
&\quad+(({}_1\boldsymbol{z}^-_n\cdot\nabla){}_1\boldsymbol{z}^+_{n+1},{}_1\boldsymbol{e}^+_{n+1})-(({}_0\boldsymbol{z}^-_n\cdot\nabla){}_0\boldsymbol{z}^+_{n+1},{}_1\boldsymbol{e}^+_{n+1})\\
&\quad+\frac{1}{2}(({}_0\boldsymbol{z}^-_{n+1}\cdot\nabla){}_0\boldsymbol{z}^+_{n+1},{}_1\boldsymbol{e}^+_{n+1})+\frac{1}{2}(({}_0\boldsymbol{z}^-_n\cdot\nabla){}_0\boldsymbol{z}^+_n,{}_1\boldsymbol{e}^+_{n+1})\Big)\\
&\quad+\Big(-\frac{1}{2}((\boldsymbol{z}^+(t_{n+1})\cdot\nabla)\boldsymbol{z}^-(t_{n+1}),{}_1\boldsymbol{e}^-_{n+1})-\frac{1}{2}((\boldsymbol{z}^+(t_n)\cdot\nabla)\boldsymbol{z}^-(t_n),{}_1\boldsymbol{e}^-_{n+1})\\
&\quad+(({}_1\boldsymbol{z}^+_n\cdot\nabla){}_1\boldsymbol{z}^-_{n+1},{}_1\boldsymbol{e}^-_{n+1})-(({}_0\boldsymbol{z}^+_n\cdot\nabla){}_0\boldsymbol{z}^-_{n+1},{}_1\boldsymbol{e}^-_{n+1})\\
&\quad+\frac{1}{2}(({}_0\boldsymbol{z}^+_{n+1}\cdot\nabla){}_0\boldsymbol{z}^-_{n+1},{}_1\boldsymbol{e}^-_{n+1})+\frac{1}{2}(({}_0\boldsymbol{z}^+_n\cdot\nabla){}_0\boldsymbol{z}^-_n,{}_1\boldsymbol{e}^-_{n+1})\Big)\\
&\quad+\Big(\frac{\nu+\nu_m}{4}(\nabla\boldsymbol{z}^+(t_{n+1}),\nabla,{}_1\boldsymbol{e}^+_{n+1})-\frac{\nu+\nu_m}{4}(\nabla\boldsymbol{z}^+(t_n),\nabla,{}_1\boldsymbol{e}^+_{n+1})\\
&\quad+\frac{\nu-\nu_m}{4}(\nabla\boldsymbol{z}^-(t_n),\nabla,{}_1\boldsymbol{e}^+_{n+1})-\frac{\nu-\nu_m}{4}(\nabla\boldsymbol{z}^-(t_{n+1}),\nabla,{}_1\boldsymbol{e}^+_{n+1})\\
&\quad-\frac{\nu+\nu_m}{4}(\nabla{}_0\boldsymbol{z}^+_{n+1},\nabla,{}_1\boldsymbol{e}^+_{n+1})-\frac{\nu-\nu_m}{4}(\nabla{}_0\boldsymbol{z}^-_n,\nabla,{}_1\boldsymbol{e}^+_{n+1})
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\nu-\nu_m}{4}(\nabla_0 \boldsymbol{z}^-_{n+1},\nabla,{}_1\boldsymbol{e}^+_{n+1})+\frac{\nu+\nu_m}{4}(\nabla_0 \boldsymbol{z}^-_{n},\nabla,{}_1\boldsymbol{e}^+_{n+1})\Big)\\
&+\Big(\frac{\nu+\nu_m}{4}(\nabla \boldsymbol{z}^-(t_{n+1}),\nabla,{}_1\boldsymbol{e}^-_{n+1})-\frac{\nu+\nu_m}{4}(\nabla \boldsymbol{z}^-(t_{n}),\nabla,{}_1\boldsymbol{e}^-_{n+1})\\
&-\frac{\nu-\nu_m}{4}(\nabla \boldsymbol{z}^+(t_{n+1}),\nabla,{}_1\boldsymbol{e}^-_{n+1})+\frac{\nu-\nu_m}{4}(\nabla \boldsymbol{z}^+(t_{n}),\nabla,{}_1\boldsymbol{e}^-_{n+1})\\
&-\frac{\nu+\nu_m}{4}(\nabla_0 \boldsymbol{z}^-_{n+1},\nabla,{}_1\boldsymbol{e}^-_{n+1})-\frac{\nu-\nu_m}{4}(\nabla_0 \boldsymbol{z}^+_{n},\nabla,{}_1\boldsymbol{e}^-_{n+1})\\
&+\frac{\nu+\nu_m}{4}(\nabla_0 \boldsymbol{z}^-_{n},\nabla,{}_1\boldsymbol{e}^-_{n+1})+\frac{\nu-\nu_m}{4}(\nabla_0 \boldsymbol{z}^-_{n+1},\nabla,{}_1\boldsymbol{e}^-_{n+1})\Big)\\
&+\Big(C\Delta t^2(\partial_{tt}\boldsymbol{F}^+,{}_1\boldsymbol{e}^+_{n+1})+C\Delta t^2(\partial_{tt}\boldsymbol{F}^-,{}_1\boldsymbol{e}^-_{n+1})\Big)\Big|.
\end{aligned}
$$

The RHS terms of (3.32) are separated into seven groups. To derive the bound for the first group we add and subtract $\frac{1}{2}((\boldsymbol{B}_\circ\cdot\nabla)\boldsymbol{z}^+(t_{n+1}),{}_1\boldsymbol{e}^+_{n+1})$ to see

$$
\begin{aligned}
&|\frac{1}{2}((\boldsymbol{B}_\circ\cdot\nabla)\boldsymbol{z}^+(t_{n+1}),{}_1\boldsymbol{e}^+_{n+1})+\frac{1}{2}((\boldsymbol{B}_\circ\cdot\nabla)\boldsymbol{z}^+(t_{n}),{}_1\boldsymbol{e}^+_{n+1})\\
&\quad-((\boldsymbol{B}_\circ\cdot\nabla)_1\boldsymbol{z}^+_{n+1},{}_1\boldsymbol{e}^+_{n+1})+\frac{1}{2}((\boldsymbol{B}_\circ\cdot\nabla)_0\boldsymbol{z}^+_{n+1},{}_1\boldsymbol{e}^+_{n+1})-\frac{1}{2}((\boldsymbol{B}_\circ\cdot\nabla)_0\boldsymbol{z}^+_{n},{}_1\boldsymbol{e}^+_{n+1})|\\
&=\frac{1}{2}((\boldsymbol{B}_\circ\cdot\nabla)({}_0\boldsymbol{e}^+_{n+1}-{}_0\boldsymbol{e}^+_{n}),{}_1\boldsymbol{e}^+_{n+1})\\
&\le C\gamma^{-1}\Delta t^2\|\nabla\boldsymbol{B}_\circ\|^2\|\nabla\boldsymbol{s}^+_{n+1}\|^2+\gamma\|\nabla_1\boldsymbol{e}^+_{n+1}\|^2.
\end{aligned}
\tag{3.33}
$$

The second group of terms in bounded as follows

$$
\begin{aligned}
&|-\frac{1}{2}((\boldsymbol{B}_\circ\cdot\nabla)\boldsymbol{z}^-(t_{n+1}),{}_1\boldsymbol{e}^-_{n+1})-\frac{1}{2}((\boldsymbol{B}_\circ\cdot\nabla)\boldsymbol{z}^-(t_{n}),{}_1\boldsymbol{e}^-_{n+1})\\
&+((\boldsymbol{B}_\circ\cdot\nabla)_1\boldsymbol{z}^-_{n+1},{}_1\boldsymbol{e}^-_{n+1})-\frac{1}{2}((\boldsymbol{B}_\circ\cdot\nabla)_0\boldsymbol{z}^-_{n+1},{}_1\boldsymbol{e}^-_{n+1})+\frac{1}{2}((\boldsymbol{B}_\circ\cdot\nabla)_0\boldsymbol{z}^-_{n},{}_1\boldsymbol{e}^-_{n+1})|\\
&\le C\gamma^{-1}\Delta t^2\|\nabla\boldsymbol{B}_\circ\|^2\|\nabla\boldsymbol{s}^-_{n+1}\|^2+\gamma\|\nabla_1\boldsymbol{e}^-_{n+1}\|^2.
\end{aligned}
\tag{3.34}
$$

The remaining grouped terms are bounded similarly and details are provided below (see A.2). Specifying $\gamma=\frac{\nu\nu_m}{26(\nu+\nu_m)}$, substituting the bounds (3.33), (3.34), (A.17), (A.18), (A.19), (A.20), (A.21), and (A.22) in to (3.32) and rearranging gives

$$
\begin{aligned}
&\frac{1}{2\Delta t}(\|{}_1\boldsymbol{e}^+_{n+1}\|^2-\|{}_1\boldsymbol{e}^+_{n}\|^2)+\frac{1}{2\Delta t}(\|{}_1\boldsymbol{e}^-_{n+1}\|^2-\|{}_1\boldsymbol{e}^-_{n}\|^2)\\
&\quad+\frac{\nu\nu_m}{2(\nu+\nu_m)}(\|\nabla_1\boldsymbol{e}^+_{n+1}\|^2+\|\nabla_1\boldsymbol{e}^-_{n+1}\|^2)\\
&\quad+\frac{(\nu-\nu_m)^2}{4(\nu+\nu_m)}(\|\nabla_1\boldsymbol{e}^+_{n+1}\|^2-\|\nabla_1\boldsymbol{e}^+_{n}\|^2+\|\nabla_1\boldsymbol{e}^-_{n+1}\|^2+\|\nabla_1\boldsymbol{e}^-_{n}\|^2)\\
&\le C\frac{\nu+\nu_m}{\nu\nu_m}\Big(\Delta t^2\|\nabla\boldsymbol{B}_\circ\|^2\|\nabla\boldsymbol{s}^+_{n+1}\|^2+\Delta t^2\|\nabla\boldsymbol{B}_\circ\|^2\|\nabla\boldsymbol{s}^-_{n+1}\|^2\\
&\quad+\Delta t^2\|\nabla\boldsymbol{z}^-(t_n)\|^2\|\nabla\boldsymbol{s}^+_{n+1}\|^2+\Delta t^2\|\nabla_0\boldsymbol{e}^-_{n}\|^2\|\nabla\boldsymbol{s}^+_{n+1}\|^2+\Delta t^2\|\nabla_0\boldsymbol{z}^-_{n}\|^2\|\nabla\boldsymbol{s}^+_{n+1}\|^2\\
&\quad+\Delta t^2\|\nabla\boldsymbol{s}^-_{n+1}\|^2\|\nabla_0\boldsymbol{e}^+_{n+1}\|^2+\Delta t^2\|\nabla\partial_t\boldsymbol{z}^-(\eta_n)\|^2\|\nabla_0\boldsymbol{e}^+_{n+1}\|^2\\
&\quad+\Delta t^2\|\nabla\boldsymbol{s}^-_{n+1}\|^2\|\nabla\boldsymbol{z}^+(t_{n+1})\|^2+\Delta t^2\|\nabla_0\boldsymbol{e}^-_{n+1}\|^2\|\nabla\partial_t\boldsymbol{z}^+(\eta_n)\|^2\\
&\quad+\Delta t^2\|\nabla\boldsymbol{s}^-_{n+1}\|^2\|\nabla\boldsymbol{z}^+(t_n)\|^2+\Delta t^2\|\nabla\boldsymbol{z}^+(t_n)\|^2\|\nabla\boldsymbol{s}^-_{n+1}\|^2\\
&\quad+\Delta t^2\|\nabla_0\boldsymbol{e}^+_{n}\|^2\|\nabla\boldsymbol{s}^-_{n+1}\|^2+\Delta t^2\|\nabla_0\boldsymbol{z}^+_{n}\|^2\|\nabla\boldsymbol{s}^-_{n+1}\|^2+\Delta t^2\|\nabla\boldsymbol{s}^+_{n+1}\|^2\|\nabla_0\boldsymbol{e}^-_{n+1}\|^2
\end{aligned}
\tag{3.35}
$$

$$
\begin{aligned}
&+\Delta t^2\|\nabla\partial_t \boldsymbol{z}^+(\eta_n)\|^2\|\nabla_0 \boldsymbol{e}^-_{n+1}\|^2+\Delta t^2\|\nabla \boldsymbol{s}^+_{n+1}\|^2\|\nabla \boldsymbol{z}^-(t_{n+1})\|^2\\
&+\Delta t^2\|\nabla_0 \boldsymbol{e}^+_{n+1}\|^2\|\nabla\partial_t \boldsymbol{z}^-(\eta_n)\|^2+\Delta t^2\|\nabla \boldsymbol{s}^+_{n+1}\|^2\|\nabla \boldsymbol{z}^-(t_n)\|^2\\
&+\Delta t^2(\nu+\nu_m)^2\|\nabla \boldsymbol{s}^+_{n+1}\|^2+\Delta t^2(\nu-\nu_m)^2\|\nabla \boldsymbol{s}^-_{n+1}\|^2+\Delta t^2(\nu+\nu_m)^2\|\nabla \boldsymbol{s}^-_{n+1}\|^2\\
&+\Delta t^2(\nu-\nu_m)^2\|\nabla \boldsymbol{s}^+_{n+1}\|^2+\Delta t^4\|\partial_{tt}\boldsymbol{F}^+(\eta_n)\|^2+\Delta t^4\|\partial_{tt}\boldsymbol{F}^-(\eta_n)\|^2\Big)\\
&+C\frac{\nu+\nu_m}{\nu\nu_m}(\|{}_1\boldsymbol{e}^-_n\|^2+\|{}_1\boldsymbol{e}^+_n\|^2)(\|\nabla \boldsymbol{z}^+(t_{n+1})\|^2_\infty+\|\nabla \boldsymbol{z}^-(t_{n+1})\|^2_\infty).
\end{aligned}
$$

Multiplying (3.35) by $2\Delta t$, summing from $n=0$ to $N-1$, and applying Gronwall's inequality finishes the proof.
□

4. Computational results. Consider a well-known test problem for the 2-D NSE: two-dimensional wave propagation (considered on a square $[0.5,1.5]\times[0.5,1.5]$). This example is chosen because the solutions are varying smoothly in space so that it is easier to track the error due to the temporal discretization; for more details on the traveling wave test problem see, e.g., [34, 26, 19] and the references therein. Let the flow be electrically conducting, and introduce the time-varying magnetic field so that the true solution (in Elsässer variables $\boldsymbol{z}^+,\boldsymbol{z}^-$) is

$$
\begin{aligned}
\boldsymbol{z}^+&=\begin{pmatrix}0.75+0.25\cos(2\pi(x-t))\sin(2\pi(y-t))e^{-8\pi^2 tRe^{-1}}+0.1(y+1)^3e^{tRe_m^{-1}}\\ -0.25\sin(2\pi(x-t))\cos(2\pi(y-t))e^{-8\pi^2 tRe^{-1}}+0.1(x+1)^3e^{tRe_m^{-1}}\end{pmatrix},\\
\boldsymbol{z}^-&=\begin{pmatrix}0.75+0.25\cos(2\pi(x-t))\sin(2\pi(y-t))e^{-8\pi^2 tRe^{-1}}-0.1(y+1)^3e^{tRe_m^{-1}}\\ -0.25\sin(2\pi(x-t))\cos(2\pi(y-t))e^{-8\pi^2 tRe^{-1}}-0.1(x+1)^3e^{tRe_m^{-1}}\end{pmatrix},\\
p&=-\frac{1}{64}(\cos(4\pi(x-t))+\cos(4\pi(y-t)))e^{-16\pi^2 tRe^{-1}}.
\end{aligned}
$$

We test the case of a laminar flow, and we also want $Re\neq Re_m$ to avoid unnecessary cancelation of terms, therefore we choose $Re=1, Re_m=10$. Galerkin finite element method is employed, using the Taylor-Hood elements (piecewise quadratic polynomials for $\boldsymbol{z}^+$ and $\boldsymbol{z}^-$ and piecewise linear polynomials for p). The results presented were obtained by using the software package $FreeFEM++$.

We compare the true solution to a solution obtained by our two-step deferred correction method. It follows from the theoretical results that an error of the order $O(k+h^2)$ is to be expected when approximating the true solution $(\boldsymbol{z}^+,\boldsymbol{z}^-)$ by the first-step variables $(w,\boldsymbol{z})$ (this is the IMEX method). Then, the correction-step variables $(cw,c\boldsymbol{z})$ should approximate the true solution $(\boldsymbol{z}^+,\boldsymbol{z}^-)$ with $O(k^2+h^2)$.

We set the time step equal to the mesh size, $k=h$, to verify the claimed second-order accuracy of the method. The tables below demonstrate the first-order accuracy of the IMEX approximation $(w,\boldsymbol{z})$ and the second-order accuracy of the correction step approximation $(cw,c\boldsymbol{z})$. The error is measured in the spatial norm $L^2([0.5,1.5]^2)$ at the final time level $T=1$, $N=\frac{T}{\Delta t}=\frac{1}{h}$.

The convergence rates in Tables 4.1, 4.2 were computed using the step sizes $h=1/4,1/16,1/64$, and the correction step approximation is clearly of the second order; if the IMEX approximations for $\boldsymbol{z}^+(T)$ and $\boldsymbol{z}^-(T)$ are computed with $h=1/64$ and $h=1/128$, then the corresponding IMEX convergence rates are 1.07 and 1.09, respectively. Hence, the computational results are consistent with the claimed accuracy of the method.

Table 4.1: First-order approximation, IMEX, $Re = 1$, $Re_m = 10$, $T = 1$.

h	$\|\boldsymbol{z}^+(T) - w^N\|_{L^2(\Omega)}$	$rate$	$\|\boldsymbol{z}^-(T) - \boldsymbol{z}^N\|_{L^2(\Omega)}$	$rate$
1/4	0.0260319		0.0240726	
1/16	0.0026907	1.64	0.00276714	1.56
1/64	0.000416796	1.35	0.000423207	1.35

Table 4.2: Correction step approximation (second order), $Re = 1$, $Re_m = 10$, $T = 1$.

h	$\|\boldsymbol{z}^+(T) - cw^N\|_{L^2(\Omega)}$	$rate$	$\|\boldsymbol{z}^-(T) - c\boldsymbol{z}^N\|_{L^2(\Omega)}$	$rate$
1/4	0.0444342		0.0466917	
1/16	0.00234416	2.12	0.00221543	2.2
1/64	0.000123658	2.12	0.00012288	2.09

5. Conclusions. When solving full evolutionary MHD systems, it is usually more computationally cheap to employ partitioned (rather than monolithic) methods. These methods aim at decoupling the MHD system by successively solving the two sub-physics problems. Not only this approach is computationally attractive (for large N it is much cheaper to solve the $N \times N$ subsystem twice, than the full $(2N) \times (2N)$ one time), but it also allows for parallelization and the use of legacy codes for the physical subproblems. An unconditionally stable (although only first-order accurate) IMEX method was proposed in [41], which decouples the full MHD system using the explicit discretization of the coupling terms. In this paper, we have introduced and thoroughly studied the higher-order accurate method, which utilizes the deferred correction approach, built on the fore-mentioned IMEX scheme. The choice of the deferred correction (as opposed to other high order methods like Adaptive Runge-Kutta or the BDFs) is based on the fact that different terms in the MHD systems can evolve on different time scales - and the deferred correction is known to be well tailored for such problems. We proved the unconditional stability of our method and (for the case of the two-step method) its second order accuracy. The claimed accuracy was then numerically verified on a test problem of the wave traveling in the presence of the magnetic field. This test problem was chosen because the error is affected mainly by its temporal component, and not the spatial component (the study of possibly different spatial discretizations is outside of the scope of this paper). As a result, we obtained a method for solving full evolutionary MHD systems in Elsässer variables, which is fast, unconditionally stable and second order accurate, and allows for the usage of different sizes of time steps for different terms of the MHD system.

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Appendix A. Appendix.

A.1. Details of proof of Lemma 3.3. The nonlinear terms of (3.19) have been separated into four groups. Treatment of the first group of nonlinear begins by applying the identity
$ab - cd = a(b-d) + (a-c)d$ twice which yields

$$\begin{aligned}
&\frac{1}{\Delta t}(({}_0\boldsymbol{z}_n^- \cdot \nabla)_0\boldsymbol{z}_{n+1}^+, \boldsymbol{s}_{n+1}^+) - \frac{1}{\Delta t}((\boldsymbol{z}^-(t_n) \cdot \nabla)\boldsymbol{z}^+(t_{n+1}), \boldsymbol{s}_{n+1}^+) \\
&\quad + \frac{1}{\Delta t}((\boldsymbol{z}^-(t_{n-1}) \cdot \nabla)\boldsymbol{z}^+(t_n), \boldsymbol{s}_{n+1}^+) - \frac{1}{\Delta t}(({}_0\boldsymbol{z}_{n-1}^- \cdot \nabla)_0\boldsymbol{z}_n^+, \boldsymbol{s}_{n+1}^+) \\
&= -\frac{1}{\Delta t}(({}_0\boldsymbol{z}_n^- \cdot \nabla)_0\boldsymbol{e}_{n+1}^+, \boldsymbol{s}_{n+1}^+) - \frac{1}{\Delta t}(({}_0\boldsymbol{e}_n^- \cdot \nabla)\boldsymbol{z}^+(t_{n+1}), \boldsymbol{s}_{n+1}^+) \\
&\quad + \frac{1}{\Delta t}((\boldsymbol{z}^-(t_{n-1}) \cdot \nabla)_0\boldsymbol{e}_n^+, \boldsymbol{s}_{n+1}^+) + \frac{1}{\Delta t}(({}_0\boldsymbol{e}_{n-1}^- \cdot \nabla)_0\boldsymbol{z}_n^+, \boldsymbol{s}_{n+1}^+).
\end{aligned} \tag{A.1}$$

Adding two zero terms to(A.1) regrouping and applying the same identity gives

$$\begin{aligned}
&\frac{1}{\Delta t}(({}_0\boldsymbol{z}_n^- \cdot \nabla)_0\boldsymbol{z}_{n+1}^+, \boldsymbol{s}_{n+1}^+) - \frac{1}{\Delta t}((\boldsymbol{z}^-(t_n) \cdot \nabla)\boldsymbol{z}^+(t_{n+1}), \boldsymbol{s}_{n+1}^+) \\
&\quad + \frac{1}{\Delta t}((\boldsymbol{z}^-(t_{n-1}) \cdot \nabla)\boldsymbol{z}^+(t_n), \boldsymbol{s}_{n+1}^+) - \frac{1}{\Delta t}(({}_0\boldsymbol{z}_{n-1}^- \cdot \nabla)_0\boldsymbol{z}_n^+, \boldsymbol{s}_{n+1}^+) \\
&= \frac{1}{\Delta t}((\boldsymbol{z}^-(t_{n-1}) \cdot \nabla)_0\boldsymbol{e}_{n+1}^+, \boldsymbol{s}_{n+1}^+) - \frac{1}{\Delta t}(({}_0\boldsymbol{z}_n^-) \cdot \nabla)_0\boldsymbol{e}_{n+1}^+, \boldsymbol{s}_{n+1}^+) \\
&\quad \pm \frac{1}{\Delta t}((\boldsymbol{z}(t_n)^- \cdot \nabla)_0\boldsymbol{e}_{n+1}^+, \boldsymbol{s}_{n+1}^+) \\
&\quad + \frac{1}{\Delta t}(({}_0\boldsymbol{e}_{n-1}^- \cdot \nabla)_0\boldsymbol{z}_n^+, \boldsymbol{s}_{n+1}^+) - \frac{1}{\Delta t}(({}_0\boldsymbol{e}_{n-1}^- \cdot \nabla)\boldsymbol{z}^+(t_{n+1}), \boldsymbol{s}_{n+1}^+)
\end{aligned} \tag{A.2}$$

$$
\begin{aligned}
&\pm \frac{1}{\Delta t}((_0\boldsymbol{e}_{n-1}^- \cdot \nabla)\boldsymbol{z}^+(t_n), \boldsymbol{s}_{n+1}^+) \\
&= \frac{1}{\Delta t}(((\boldsymbol{z}^-(t_{n-1}) - \boldsymbol{z}^-(t_n)) \cdot \nabla)_0\boldsymbol{e}_{n+1}^+, \boldsymbol{s}_{n+1}^+) + ((_0s_n^-) \cdot \nabla)_0\boldsymbol{e}_{n+1}^+, \boldsymbol{s}_{n+1}^+) \\
&\quad + \frac{1}{\Delta t}((_0\boldsymbol{e}_{n-1}^- \cdot \nabla)(\boldsymbol{z}^+(t_n) - \boldsymbol{z}^+(t_{n+1})), \boldsymbol{s}_{n+1}^+) - ((\boldsymbol{s}_n^- \cdot \nabla)\boldsymbol{z}^+(t_{n+1}), \boldsymbol{s}_{n+1}^+).
\end{aligned}
$$

We now bound the terms in (A.2) with standard inequalities as follows

$$
\begin{aligned}
&\frac{1}{\Delta t}|(((\boldsymbol{z}^-(t_{n-1} - \boldsymbol{z}^-(t_n)) \cdot \nabla)_0\boldsymbol{e}_{n+1}^+, \boldsymbol{s}_{n+1}^+)| \qquad \text{(A.3)} \\
&\quad \le C\|\partial_t\boldsymbol{z}^-(\eta_n)\|_\infty\|\nabla_0\boldsymbol{e}_{n+1}^+\|\|\nabla\boldsymbol{s}_{n+1}^+\| \\
&\quad \le C\gamma^{-1}\|\partial_t\boldsymbol{z}^-(\eta_n)\|_\infty^2\|\nabla_0\boldsymbol{e}_{n+1}^+\|^2 + \gamma\|\nabla\boldsymbol{s}_{n+1}^+\|^2, \\
&|((_0s_n^-) \cdot \nabla)_0\boldsymbol{e}_{n+1}^+, \boldsymbol{s}_{n+1}^+)| \le C\|_0s_n^-\|^{\frac{1}{2}}\|\nabla_0s_n^-\|^{\frac{1}{2}}\|\nabla_0\boldsymbol{e}_{n+1}^+\|\|\nabla\boldsymbol{s}_{n+1}^+\| \qquad \text{(A.4)} \\
&\quad \le C\gamma^{-1}C\|_0s_n^-\|\|\nabla_0s_n^-\|\|\nabla_0\boldsymbol{e}_{n+1}^+\|^2 + \gamma\|\nabla\boldsymbol{s}_{n+1}^+\|^2 \\
&\qquad \le C\gamma^{-1}\|_0s_n^-\|^2 + C\gamma^{-1}\|\nabla_0s_n^-\|^2\|\nabla_0\boldsymbol{e}_{n+1}^+\|^4 + \gamma\|\nabla\boldsymbol{s}_{n+1}^+\|^2, \\
&\frac{1}{\Delta t}|((_0\boldsymbol{e}_{n-1}^- \cdot \nabla)(\boldsymbol{z}^+(t_n) - \boldsymbol{z}^+(t_{n+1})), \boldsymbol{s}_{n+1}^+)| \qquad \text{(A.5)} \\
&\quad \le C\|_0\boldsymbol{e}_{n-1}^-\|\|\nabla\partial_t\boldsymbol{z}^+(\eta_n)\|_\infty\|\nabla\boldsymbol{s}_{n+1}^+\| \\
&\quad \le C\gamma^{-1}\|_0\boldsymbol{e}_{n-1}^-\|^2\|\nabla\partial_t\boldsymbol{z}^+(\eta_n)\|_\infty^2 + \gamma\|\nabla\boldsymbol{s}_{n+1}^+\|^2, \\
&|((\boldsymbol{s}_n^- \cdot \nabla)\boldsymbol{z}^+(t_{n+1}), \boldsymbol{s}_{n+1}^+)| \qquad \text{(A.6)} \\
&\le \|\boldsymbol{s}_n^-\|\|\nabla\boldsymbol{z}^+(t_{n+1})\|_\infty\|\nabla\boldsymbol{s}_{n+1}^+\| \le C\gamma^{-1}\|\boldsymbol{s}_n^-\|^2\|\nabla\boldsymbol{z}^+(t_{n+1})\|_\infty^2 + \gamma\|\nabla\boldsymbol{s}_{n+1}^+\|^2.
\end{aligned}
$$

Combining (A.3)-(A.6) gives the following bound for the first group of nonlinear terms

$$
\begin{aligned}
&\Big|\frac{1}{\Delta t}|((_0\boldsymbol{z}_n^- \cdot \nabla)_0\boldsymbol{z}_{n+1}^+, \boldsymbol{s}_{n+1}^+) - \frac{1}{\Delta t}((\boldsymbol{z}^-(t_n) \cdot \nabla)\boldsymbol{z}^+(t_{n+1}), \boldsymbol{s}_{n+1}^+) \qquad \text{(A.7)} \\
&\qquad + \frac{1}{\Delta t}((\boldsymbol{z}^-(t_{n-1}) \cdot \nabla)\boldsymbol{z}^+(t_n), \boldsymbol{s}_{n+1}^+) - \frac{1}{\Delta t}((_0\boldsymbol{z}_{n-1}^- \cdot \nabla)_0\boldsymbol{z}_n^+, \boldsymbol{s}_{n+1}^+)\Big| \\
&\le C\gamma^{-1}\Big(\|\partial_t\boldsymbol{z}^-(\eta_n)\|_\infty^2\|\nabla_0\boldsymbol{e}_{n+1}^+\|^2 + \|_0s_n^-\|^2 + \|\nabla s_n^-\|^2\|\nabla_0\boldsymbol{e}_{n+1}^+\|^4 \\
&\qquad \|_0\boldsymbol{e}_{n-1}^-\|^2\|\nabla\partial_t\boldsymbol{z}^+(\eta_n)\|_\infty^2 + \|\boldsymbol{s}_n^-\|^2\|\nabla\boldsymbol{z}^+(t_{n+1})\|_\infty^2\Big) + 4\gamma\|\nabla s_{n+1}^+\|^2.
\end{aligned}
$$

Similar treatment of the second group of nonlinear terms yields the following bound

$$
\begin{aligned}
&\Big|\frac{1}{\Delta t}((_0\boldsymbol{z}_n^+ \cdot \nabla)_0\boldsymbol{z}_{n+1}^-, s_{n+1}^-) - \frac{1}{\Delta t}((\boldsymbol{z}^+(t_n) \cdot \nabla)\boldsymbol{z}^-(t_{n+1}), s_{n+1}^-) \qquad \text{(A.8)} \\
&\qquad + \frac{1}{\Delta t}((\boldsymbol{z}^+(t_{n-1}) \cdot \nabla)\boldsymbol{z}^-(t_n), s_{n+1}^-) - \frac{1}{\Delta t}((_0\boldsymbol{z}_{n-1}^+ \cdot \nabla)_0\boldsymbol{z}_n^-, \boldsymbol{s}_{n+1}^-)\Big| \\
&\le C\gamma^{-1}\Big(\|\partial_t\boldsymbol{z}^+(\eta_n)\|_\infty^2\|\nabla_0\boldsymbol{e}_{n+1}^-\|^2 + \|_0s_n^+\|^2 + \|\nabla s_n^+\|^2\|\nabla_0\boldsymbol{e}_{n+1}^-\|^4 \\
&\qquad + \|_0\boldsymbol{e}_{n-1}^+\|^2\|\nabla\partial_t\boldsymbol{z}^-(\eta_n)\|_\infty^2 + \|\boldsymbol{s}_n^+\|^2\|\nabla\boldsymbol{z}^-(t_{n+1})\|_\infty^2\Big) + 4\gamma\|\nabla\boldsymbol{s}_{n+1}^-\|^2.
\end{aligned}
$$

We now derive bounds for the third group of nonlinear terms. Grouping the terms linearly, applying the identity ab-cd = a(b-d)+(a-c)d, and standard bounds on the nonlinearity gives

$$
\Big|\frac{1}{\Delta t}((\boldsymbol{z}^-(t_n) \cdot \nabla)\boldsymbol{z}^+(t_{n+1}), \boldsymbol{s}_{n+1}^+) - \frac{1}{\Delta t}((\boldsymbol{z}^-(t_{n+1}) \cdot \nabla)\boldsymbol{z}^+(t_{n+1}), \boldsymbol{s}_{n+1}^+) \qquad \text{(A.9)}
$$

$$
\begin{aligned}
&+\frac{1}{\Delta t}((\boldsymbol{z}^-(t_n)\cdot\nabla)\boldsymbol{z}^+(t_n),\boldsymbol{s}^+_{n+1})-\frac{1}{\Delta t}((\boldsymbol{z}^-(t_{n-1})\cdot\nabla)\boldsymbol{z}^+(t_n),\boldsymbol{s}^+_{n+1})\Big|\\
&\le C\gamma^{-1}\Delta t^2\|\nabla\partial_t\boldsymbol{z}^-(\eta_n)\|^2\|\nabla\partial_t\boldsymbol{z}^+(\eta_n)\|^2\\
&\quad+C\gamma^{-1}\Delta t^2\|\nabla\partial_{tt}\boldsymbol{z}^-(\eta_n)\|^2\|\nabla\boldsymbol{z}^+(t_n)\|^2+2\gamma\|\nabla\boldsymbol{s}^+_{n+1}\|^2.
\end{aligned}
$$

Similar treatment of the fourth group of nonlinear terms gives

$$
\begin{aligned}
&\Big|\frac{1}{\Delta t}((\boldsymbol{z}^+(t_n)\cdot\nabla)\boldsymbol{z}^-(t_{n+1}),\boldsymbol{s}^-_{n+1})-\frac{1}{\Delta t}((\boldsymbol{z}^+(t_{n+1})\cdot\nabla)\boldsymbol{z}^-(t_{n+1}),\boldsymbol{s}^-_{n+1})\\
&\quad+\frac{1}{\Delta t}((\boldsymbol{z}^+(t_n)\cdot\nabla)\boldsymbol{z}^-(t_n),\boldsymbol{s}^-_{n+1})-\frac{1}{\Delta t}((\boldsymbol{z}^+(t_{n-1})\cdot\nabla)\boldsymbol{z}^-(t_n),\boldsymbol{s}^-_{n+1})\Big|\\
&\le C\gamma^{-1}\Delta t^2\|\nabla\partial_t\boldsymbol{z}^+(\eta_n)\|^2\|\nabla\partial_t\boldsymbol{z}^-(\eta_n)\|^2\\
&\quad+C\gamma^{-1}\Delta t^2\|\nabla\partial_{tt}\boldsymbol{z}^+(\eta_n)\|^2\|\nabla\boldsymbol{z}^-(t_n)\|^2+2\gamma\|\nabla\boldsymbol{s}^-_{n+1}\|^2.
\end{aligned}
\tag{A.10}
$$

A.2. Details of Proof for Theorem 3.4. To bound the third group of terms in (3.32) we add and subtract the term $((\boldsymbol{z}^-(t_n)\cdot\nabla)\boldsymbol{z}^+(t_{n+1}),{}_1\boldsymbol{e}^+_{n+1})$ to get

$$
\begin{aligned}
&((_1\boldsymbol{z}^-_n\cdot\nabla)_1\boldsymbol{z}^+_{n+1},{}_1\boldsymbol{e}^+_{n+1})-((\boldsymbol{z}^-(t_n)\cdot\nabla)\boldsymbol{z}^+(t_{n+1}),{}_1\boldsymbol{e}^+_{n+1})\\
&+((\boldsymbol{z}^-(t_n)\cdot\nabla)\boldsymbol{z}^+(t_{n+1}),{}_1\boldsymbol{e}^+_{n+1})-((_0\boldsymbol{z}^-_n\cdot\nabla)_0\boldsymbol{z}^+_{n+1},{}_1\boldsymbol{e}^+_{n+1})\\
&+\frac{1}{2}((_0\boldsymbol{z}^-_{n+1}\cdot\nabla)_0\boldsymbol{z}^+_{n+1},{}_1\boldsymbol{e}^+_{n+1})-\frac{1}{2}((\boldsymbol{z}^-(t_{n+1})\cdot\nabla)\boldsymbol{z}^+(t_{n+1}),{}_1\boldsymbol{e}^+_{n+1})\\
&+\frac{1}{2}((_0\boldsymbol{z}^-_n\cdot\nabla)_0\boldsymbol{z}^+_n,{}_1\boldsymbol{e}^+_{n+1})-\frac{1}{2}((\boldsymbol{z}^-(t_n)\cdot\nabla)\boldsymbol{z}^+(t_n),{}_1\boldsymbol{e}^+_{n+1}).
\end{aligned}
\tag{A.11}
$$

Using the identity $ab-cd=a(b-d)+(a-c)d$ on the terms of (A.11) gives

$$
\begin{aligned}
&0-((_1\boldsymbol{e}^-_n\cdot\nabla)\boldsymbol{z}^+(t_{n+1}),{}_1\boldsymbol{e}^+_{n+1})\\
&+((\boldsymbol{z}^-(t_n)\cdot\nabla)_0\boldsymbol{e}^+_{n+1},{}_1\boldsymbol{e}^+_{n+1})+((_0\boldsymbol{e}^-_n\cdot\nabla)_0\boldsymbol{z}^+_{n+1},{}_1\boldsymbol{e}^+_{n+1})\\
&-\frac{1}{2}((_0\boldsymbol{z}^-_{n+1}\cdot\nabla)_0\boldsymbol{e}^+_{n+1},{}_1\boldsymbol{e}^+_{n+1})-\frac{1}{2}((_0\boldsymbol{e}^-_{n+1}\cdot\nabla)\boldsymbol{z}^+(t_{n+1}),{}_1\boldsymbol{e}^+_{n+1})\\
&-\frac{1}{2}((_0\boldsymbol{z}^-_n\cdot\nabla)_0\boldsymbol{e}^+_n,{}_1\boldsymbol{e}^+_{n+1})-\frac{1}{2}((_0\boldsymbol{e}^-_n\cdot\nabla)\boldsymbol{z}^+(t_n),{}_1\boldsymbol{e}^+_{n+1}).
\end{aligned}
\tag{A.12}
$$

We continue to derive the bound for the third group of nonlinear terms by focussing on the third through eighth terms in (A.12). Combining the third, fifth, and seventh terms in (A.12) and adding and subtracting $\frac{1}{2}((_0\boldsymbol{z}^-_n\cdot\nabla)_0\boldsymbol{e}^+_n,{}_1\boldsymbol{e}^+_{n+1})$ gives

$$
\begin{aligned}
&((\boldsymbol{z}^-(t_n)\cdot\nabla)_0\boldsymbol{e}^+_{n+1},{}_1\boldsymbol{e}^+_{n+1})-((_0\boldsymbol{z}^-_n\cdot\nabla)_0\boldsymbol{e}^+_n,{}_1\boldsymbol{e}^+_{n+1})\\
&\quad+\frac{1}{2}((_0\boldsymbol{z}^-_n\cdot\nabla)_0\boldsymbol{e}^+_n,{}_1\boldsymbol{e}^+_{n+1})-\frac{1}{2}((_0\boldsymbol{z}^-_{n+1}\cdot\nabla)_0\boldsymbol{e}^+_{n+1},{}_1\boldsymbol{e}^+_{n+1})\\
&=((\boldsymbol{z}^-(t_n)\cdot\nabla)(_0\boldsymbol{e}^+_{n+1}-{}_0\boldsymbol{e}^+_n),{}_1\boldsymbol{e}^+_{n+1})+((_0\boldsymbol{e}^-_n\cdot\nabla)_0\boldsymbol{e}^+_n,{}_1\boldsymbol{e}^+_{n+1})\\
&\quad+\frac{1}{2}((_0\boldsymbol{z}^-_n\cdot\nabla)(_0\boldsymbol{e}^+_n-{}_0\boldsymbol{e}^+_{n+1}),{}_1\boldsymbol{e}^+_{n+1})+\frac{1}{2}(((_0\boldsymbol{z}^-_n-{}_0\boldsymbol{z}^-_{n+1})\cdot\nabla)_0\boldsymbol{e}^+_{n+1},{}_1\boldsymbol{e}^+_{n+1}).
\end{aligned}
\tag{A.13}
$$

The last term in (A.13) requires further attention

$$
\begin{aligned}
&\frac{1}{2}(((_0\boldsymbol{z}^-_n-{}_0\boldsymbol{z}^-_{n+1})\cdot\nabla)_0\boldsymbol{e}^+_n,{}_1\boldsymbol{e}^+_{n+1})\pm\frac{1}{2}(((\boldsymbol{z}^-(t_n)-\boldsymbol{z}^-(t_{n+1}))\cdot\nabla,{}_0\boldsymbol{e}^+_{n+1},{}_1\boldsymbol{e}^+_{n+1})\\
&=\frac{1}{2}(((_0\boldsymbol{e}^-_{n+1}-{}_0\boldsymbol{e}^-_n)\cdot\nabla)_0\boldsymbol{e}^+_{n+1},{}_1\boldsymbol{e}^+_{n+1})
\end{aligned}
\tag{A.14}
$$

$$+\frac{1}{2}(((\boldsymbol{z}^-(t_n)-\boldsymbol{z}^-(t_{n+1}))\cdot\nabla),{}_0\boldsymbol{e}^+_{n+1},{}_1\boldsymbol{e}^+_{n+1}).$$

Next, combining the fourth, sixth, and eighth terms in (A.12) and adding and subtracting $\frac{1}{2}(({}_0\boldsymbol{e}^-_{n+1}\cdot\nabla){}_0\boldsymbol{z}^+(t_{n+1}),{}_1\boldsymbol{e}^+_{n+1})$ gives

$$\begin{aligned}
&(({}_0\boldsymbol{e}^-_n\cdot\nabla){}_0\boldsymbol{z}^+_{n+1},{}_1\boldsymbol{e}^+_{n+1})-(({}_0\boldsymbol{e}^-_{n+1}\cdot\nabla)\boldsymbol{z}^+(t_{n+1}),{}_1\boldsymbol{e}^+_{n+1}) \\
&\quad+\frac{1}{2}(({}_0\boldsymbol{e}^-_{n+1}\cdot\nabla){}_0\boldsymbol{z}^+(t_{n+1}),{}_1\boldsymbol{e}^+_{n+1})-\frac{1}{2}(({}_0\boldsymbol{e}^-_n\cdot\nabla){}_0\boldsymbol{z}^+(t_n),{}_1\boldsymbol{e}^+_{n+1}) \\
&=-(({}_0\boldsymbol{e}^-_n\cdot\nabla){}_0\boldsymbol{e}^+_{n+1},{}_1\boldsymbol{e}^+_{n+1})+(({}_0\boldsymbol{e}^-_n-{}_0\boldsymbol{e}^-_{n+1})\cdot\nabla)\boldsymbol{z}^+(t_{n+1}),{}_1\boldsymbol{e}^+_{n+1}) \\
&\quad+\frac{1}{2}(({}_0\boldsymbol{e}^-_{n+1}\cdot\nabla)(\boldsymbol{z}^+(t_{n+1})-\boldsymbol{z}^+(t_n)),{}_1\boldsymbol{e}^+_{n+1}) \\
&\quad+\frac{1}{2}(({}_0\boldsymbol{e}^-_{n+1}-{}_0\boldsymbol{e}^-_n)\cdot\nabla)\boldsymbol{z}^+(t_n),{}_1\boldsymbol{e}^+_{n+1}).
\end{aligned}\tag{A.15}$$

Substituting, (A.13)-(A.15) gives the following equality

$$\begin{aligned}
&-\frac{1}{2}((\boldsymbol{z}^-(t_{n+1})\cdot\nabla)\boldsymbol{z}^+(t_{n+1}),{}_1\boldsymbol{e}^+_{n+1})-\frac{1}{2}((\boldsymbol{z}^-(t_n)\cdot\nabla)\boldsymbol{z}^+(t_n),{}_1\boldsymbol{e}^+_{n+1}) \\
&\quad+(({}_1\boldsymbol{z}^-_n\cdot\nabla){}_1\boldsymbol{z}^+_{n+1},{}_1\boldsymbol{e}^+_{n+1})-(({}_0\boldsymbol{z}^-_n\cdot\nabla){}_0\boldsymbol{z}^+_{n+1},{}_1\boldsymbol{e}^+_{n+1}) \\
&\quad+\frac{1}{2}(({}_0\boldsymbol{z}^-_{n+1}\cdot\nabla){}_0\boldsymbol{z}^+_{n+1},{}_1\boldsymbol{e}^+_{n+1})+\frac{1}{2}(({}_0\boldsymbol{z}^-_n\cdot\nabla){}_0\boldsymbol{z}^+_n,{}_1\boldsymbol{e}^+_{n+1}) \\
&=(({}_1\boldsymbol{e}^-_n\cdot\nabla)\boldsymbol{z}^+(t_{n+1}),{}_1\boldsymbol{e}^+_{n+1})+((\boldsymbol{z}^-(t_n)\cdot\nabla)({}_0\boldsymbol{e}^+_{n+1}-{}_0\boldsymbol{e}^+_n),{}_1\boldsymbol{e}^+_{n+1}) \\
&\quad+(({}_0\boldsymbol{e}^-_n\cdot\nabla)({}_0\boldsymbol{e}^+_n-{}_0\boldsymbol{e}^+_{n+1}),{}_1\boldsymbol{e}^+_{n+1})+\frac{1}{2}(({}_0\boldsymbol{z}^-_n\cdot\nabla)({}_0\boldsymbol{e}^+_n-{}_0\boldsymbol{e}^+_{n+1}),{}_1\boldsymbol{e}^+_{n+1}) \\
&\quad+\frac{1}{2}(({}_0\boldsymbol{e}^-_{n+1}-{}_0\boldsymbol{e}^-_n)\cdot\nabla){}_0\boldsymbol{e}^+_{n+1},{}_1\boldsymbol{e}^+_{n+1})+\frac{1}{2}(({}_0\boldsymbol{e}^-_{n+1}-{}_0\boldsymbol{e}^-_n)\cdot\nabla)\boldsymbol{z}^+(t_n),{}_1\boldsymbol{e}^+_{n+1}) \\
&\quad+(({}_0\boldsymbol{e}^-_n-{}_0\boldsymbol{e}^-_{n+1})\cdot\nabla)\boldsymbol{z}^+(t_{n+1}),{}_1\boldsymbol{e}^+_{n+1}) \\
&\quad+\frac{1}{2}(({}_0\boldsymbol{e}^-_{n+1}\cdot\nabla)(\boldsymbol{z}^+(t_{n+1})-\boldsymbol{z}^+(t_n)),{}_1\boldsymbol{e}^+_{n+1}) \\
&\quad+\frac{1}{2}(((\boldsymbol{z}^-(t_n)-\boldsymbol{z}^-(t_{n+1}))\cdot\nabla),{}_0\boldsymbol{e}^+_{n+1},{}_1\boldsymbol{e}^+_{n+1}).
\end{aligned}\tag{A.16}$$

Applying standard inequalities gives the following upperbound for the third group of nonlinear terms in (3.32)

$$\begin{aligned}
&\Big|-\frac{1}{2}((\boldsymbol{z}^-(t_{n+1})\cdot\nabla)\boldsymbol{z}^+(t_{n+1}),{}_1\boldsymbol{e}^+_{n+1})-\frac{1}{2}((\boldsymbol{z}^-(t_n)\cdot\nabla)\boldsymbol{z}^+(t_n),{}_1\boldsymbol{e}^+_{n+1}) \\
&\quad+(({}_1\boldsymbol{z}^-_n\cdot\nabla){}_1\boldsymbol{z}^+_{n+1},{}_1\boldsymbol{e}^+_{n+1})-(({}_0\boldsymbol{z}^-_n\cdot\nabla){}_0\boldsymbol{z}^+_{n+1},{}_1\boldsymbol{e}^+_{n+1}) \\
&\quad+\frac{1}{2}(({}_0\boldsymbol{z}^-_{n+1}\cdot\nabla){}_0\boldsymbol{z}^+_{n+1},{}_1\boldsymbol{e}^+_{n+1})+\frac{1}{2}(({}_0\boldsymbol{z}^-_n\cdot\nabla){}_0\boldsymbol{z}^+_n,{}_1\boldsymbol{e}^+_{n+1})\Big| \\
&\le C\gamma^{-1}\|{}_1\boldsymbol{e}^-_n\|^2\|\nabla\boldsymbol{z}^+(t_{n+1})\|^2_\infty+C\gamma^{-1}\Delta t^2\Big(\|\nabla\boldsymbol{z}^-(t_n)\|^2\|\nabla\boldsymbol{s}^+_{n+1}\|^2 \\
&\quad+\|\nabla{}_0\boldsymbol{e}^-_n\|^2\|\nabla\boldsymbol{s}^+_{n+1}\|^2+\|\nabla{}_0\boldsymbol{z}^-_n\|^2\|\nabla\boldsymbol{s}^+_{n+1}\|^2+\|\nabla\boldsymbol{s}^-_{n+1}\|^2\|\nabla{}_0\boldsymbol{e}^+_{n+1}\|^2 \\
&\quad+\|\nabla\partial_t\boldsymbol{z}^-(\eta_n)\|^2\|\nabla{}_0\boldsymbol{e}^+_{n+1}\|^2+\|\nabla\boldsymbol{s}^-_{n+1}\|^2\|\nabla\boldsymbol{z}^+(t_{n+1})\|^2 \\
&\quad+\|\nabla{}_0\boldsymbol{e}^-_{n+1}\|^2\|\nabla\partial_t\boldsymbol{z}^+(\eta_n)\|^2+\|\nabla\boldsymbol{s}^-_{n+1}\|^2\|\nabla\boldsymbol{z}^+(t_n)\|^2\Big)+9\gamma\|\nabla{}_1\boldsymbol{e}^+_{n+1}\|^2.
\end{aligned}\tag{A.17}$$

The bound for the fourth group of terms in (3.32) is derived in a similar way. The bound is

$$
\begin{aligned}
&\Big| -\frac{1}{2}((\boldsymbol{z}^+(t_{n+1})\cdot\nabla)\boldsymbol{z}^-(t_{n+1}),{}_1\boldsymbol{e}^-_{n+1}) - \frac{1}{2}((\boldsymbol{z}^+(t_n)\cdot\nabla)\boldsymbol{z}^-(t_n),{}_1\boldsymbol{e}^-_{n+1}) \\
&\quad + (({}_1\boldsymbol{z}^+_n\cdot\nabla){}_1\boldsymbol{z}^-_{n+1},{}_1\boldsymbol{e}^-_{n+1}) - (({}_0\boldsymbol{z}^+_n\cdot\nabla){}_0\boldsymbol{z}^-_{n+1},{}_1\boldsymbol{e}^-_{n+1}) \\
&\quad + \frac{1}{2}(({}_0\boldsymbol{z}^+_{n+1}\cdot\nabla){}_0\boldsymbol{z}^-_{n+1},{}_1\boldsymbol{e}^-_{n+1}) + \frac{1}{2}(({}_0\boldsymbol{z}^+_n\cdot\nabla){}_0\boldsymbol{z}^-_n,{}_1\boldsymbol{e}^-_{n+1})\Big| \\
&\le C\gamma^{-1}\|{}_1\boldsymbol{e}^+_n\|^2\|\nabla\boldsymbol{z}^-(t_{n+1})\|^2_\infty + C\gamma^{-1}\Delta t^2\Big(\|\nabla\boldsymbol{z}^+(t_n)\|^2\|\nabla\boldsymbol{s}^-_{n+1}\|^2 \\
&\quad + \|\nabla{}_0\boldsymbol{e}^+_n\|^2\|\nabla\boldsymbol{s}^-_{n+1}\|^2 + \|\nabla{}_0\boldsymbol{z}^+_n\|^2\|\nabla\boldsymbol{s}^-_{n+1}\|^2 + \|\nabla\boldsymbol{s}^+_{n+1}\|^2\|\nabla{}_0\boldsymbol{e}^-_{n+1}\|^2 \\
&\quad + \|\nabla\partial_t\boldsymbol{z}^+(\eta_n)\|^2\|\nabla{}_0\boldsymbol{e}^-_{n+1}\|^2 + \|\nabla\boldsymbol{s}^+_{n+1}\|^2\|\nabla\boldsymbol{z}^-(t_{n+1})\|^2 \\
&\quad + \|\nabla{}_0\boldsymbol{e}^+_{n+1}\|^2\|\nabla\partial_t\boldsymbol{z}^-(\eta_n)\|^2 + \|\nabla\boldsymbol{s}^+_{n+1}\|^2\|\nabla\boldsymbol{z}^-(t_n)\|^2\Big) + 9\gamma\|\nabla{}_1\boldsymbol{e}^-_{n+1}\|^2.
\end{aligned}
\tag{A.18}
$$

Combing the terms in the fifth group of terms in (3.32) and applying standard inequalities gives

$$
\begin{aligned}
&\Big|\frac{\nu+\nu_m}{4}(\nabla\boldsymbol{z}^+(t_{n+1}),\nabla,{}_1\boldsymbol{e}^+_{n+1}) - \frac{\nu+\nu_m}{4}(\nabla{}_0\boldsymbol{z}^+_{n+1},\nabla,{}_1\boldsymbol{e}^+_{n+1}) \\
&\quad + \frac{\nu-\nu_m}{4}(\nabla\boldsymbol{z}^-(t_n),\nabla,{}_1\boldsymbol{e}^+_{n+1}) - \frac{\nu-\nu_m}{4}(\nabla{}_0\boldsymbol{z}^-_n,\nabla,{}_1\boldsymbol{e}^+_{n+1}) \\
&\quad + \frac{\nu+\nu_m}{4}(\nabla{}_0\boldsymbol{z}^-_n,\nabla,{}_1\boldsymbol{e}^+_{n+1}) - \frac{\nu+\nu_m}{4}(\nabla\boldsymbol{z}^+(t_n),\nabla,{}_1\boldsymbol{e}^+_{n+1}) \\
&\quad + \frac{\nu-\nu_m}{4}(\nabla{}_0\boldsymbol{z}^-_{n+1},\nabla,{}_1\boldsymbol{e}^+_{n+1}) - \frac{\nu-\nu_m}{4}(\nabla\boldsymbol{z}^-(t_{n+1}),\nabla,{}_1\boldsymbol{e}^+_{n+1})\Big| \\
&\le C\Delta t^2\gamma^{-1}(\nu+\nu_m)^2\|\nabla\boldsymbol{s}^+_{n+1}\|^2 + C\Delta t^2\gamma^{-1}(\nu-\nu_m)^2\|\nabla\boldsymbol{s}^-_{n+1}\|^2 + 2\gamma\|{}_1\boldsymbol{e}^+_{n+1}\|^2.
\end{aligned}
\tag{A.19}
$$

The sixth group of terms in (3.32) are bounded similarly

$$
\begin{aligned}
&\Big|\frac{\nu+\nu_m}{4}(\nabla\boldsymbol{z}^-(t_{n+1}),\nabla,{}_1\boldsymbol{e}^-_{n+1}) - \frac{\nu+\nu_m}{4}(\nabla{}_0\boldsymbol{z}^-_{n+1},\nabla,{}_1\boldsymbol{e}^-_{n+1}) \\
&\quad + \frac{\nu+\nu_m}{4}(\nabla{}_0\boldsymbol{z}^-_n,\nabla,{}_1\boldsymbol{e}^-_{n+1}) - \frac{\nu+\nu_m}{4}(\nabla\boldsymbol{z}^-(t_n),\nabla,{}_1\boldsymbol{e}^-_{n+1}) \\
&\quad + \frac{\nu-\nu_m}{4}(\nabla{}_0\boldsymbol{z}^+_{n+1},\nabla,{}_1\boldsymbol{e}^-_{n+1}) - \frac{\nu-\nu_m}{4}(\nabla\boldsymbol{z}^+(t_{n+1}),\nabla,{}_1\boldsymbol{e}^-_{n+1}) \\
&\quad + \frac{\nu-\nu_m}{4}(\nabla\boldsymbol{z}^+(t_n),\nabla,{}_1\boldsymbol{e}^-_{n+1}) - \frac{\nu-\nu_m}{4}(\nabla{}_0\boldsymbol{z}^+_n,\nabla,{}_1\boldsymbol{e}^-_{n+1})\Big| \\
&\le C\Delta t^2\gamma^{-1}(\nu+\nu_m)^2\|\nabla\boldsymbol{s}^-_{n+1}\|^2 + C\Delta t^2\gamma^{-1}(\nu-\nu_m)^2\|\nabla\boldsymbol{s}^+_{n+1}\|^2 + 2\gamma\|{}_1\boldsymbol{e}^-_{n+1}\|^2.
\end{aligned}
\tag{A.20}
$$

The remaining two terms in (3.32) are bounded as follows

$$
\Delta t^2|(\partial_{tt}\boldsymbol{F}^+,{}_1\boldsymbol{e}^+_{n+1})| \le C\Delta t^4\gamma^{-1}\|\partial_{tt}\boldsymbol{F}^+(\eta_n)\|^2 + \gamma\|\nabla{}_1\boldsymbol{e}^+_{n+1}\|^2, \tag{A.21}
$$

$$
\Delta t^2|(\partial_{tt}\boldsymbol{F}^-,{}_1\boldsymbol{e}^-_{n+1})| \le C\Delta t^4\gamma^{-1}\|\partial_{tt}\boldsymbol{F}^-(\eta_n)\|^2 + \gamma\|\nabla{}_1\boldsymbol{e}^-_{n+1}\|^2. \tag{A.22}
$$