Abstract: We introduce some basic examples which lead to the study of stochastic homogenization. Then we introduce the minimal energy ν of the dirichlet problem

$$\begin{cases} -\operatorname{div}(\mathbf{a}\nabla u) &= 0 & \text{in } U, \\ u &= \ell_p & \text{on } \partial U, \end{cases}$$

for suitable U and $\mathbf{a}$. We show basic properties of ν and the minimizer v and show convergence in some appropriate senses. As we will see, both quantities give us important information about our problem.

1 Notation

Throughout every talk, we fix the constants $d \geq 2$ and $\Lambda > 1$. We define the *state space*

$$\Omega \coloneqq \left\{ \mathbf{a} \colon \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times d}_{\text{sym}} \colon \mathbf{a} \text{ is measurable, and } |\xi|^2 \leq \xi \cdot \mathbf{a}\xi \leq \Lambda|\xi|^2 \text{ for all } \xi \in \mathbb{R}^d \right\}. \tag{1.1}$$

The entries of a matrix $\mathbf{a} \in \Omega$ are denoted by $\mathbf{a} = \{a_{i,j}\}_{i,j\in\{1,\ldots,d\}}$. Let $U \subset \mathbb{R}^d$ be a Borel set, then we define $\mathcal{F}_U$ to be the smallest σ-Algebra on Ω, such that the mappings

$$\mathbf{a} \longmapsto \int_U a_{i,j}(x)\,\varphi(x)\,\mathrm{d}x \qquad \text{are measurable for all } i,j \in \{1,\ldots,d\} \text{ and } \varphi \in C_c^\infty(U). \tag{1.2}$$

For simplicity, we write $\mathcal{F} \coloneqq \mathcal{F}_{\mathbb{R}^d}$. By definition, $(\Omega, \mathcal{F})$ is a measurable space.

Let $y \in \mathbb{R}^d$, then we define the *translation operator* T_y by

$$\mathbf{a} \circ T_y(x) = \mathbf{a}(x+y) \qquad \text{for all } x \in \mathbb{R}^d. \tag{1.3}$$

We denote by $\mathbb{P}$ a probability measure on $(\Omega, \mathcal{F})$, which satisfies the following two conditions:

- $\mathbb{Z}^d$-*stationarity*: For every $z \in \mathbb{Z}^d$, we have
$$\mathbb{P} \circ T_z = \mathbb{P}. \tag{1.4}$$
- *Unit range of dependence*: For every pair of Borel sets $U, V \subset \mathbb{R}^d$ with $\operatorname{dist}(U,V) \geq 1$, the σ-Algebras $\mathcal{F}_U$ and $\mathcal{F}_V$ are $\mathbb{P}$-independent.

For any random variable $X \colon \Omega \longrightarrow \mathbb{R}$, we denote the *expectation with respect to* $\mathbb{P}$ by $\mathbb{E}$, that is

$$\mathbb{E}[X] = \int_\Omega X(\mathbf{a})\,\mathrm{d}\mathbb{P}(\mathbf{a}) = \int_\Omega X\,\mathrm{d}\mathbb{P}. \tag{1.5}$$

For simplicity, we do not always state the dependence on $\mathbf{a}$. Let $Y \colon \Omega \longrightarrow \mathbb{R}$ be another random variable. We also define the *variance* resp. *covariance* by

$$\operatorname{var}(X) = \mathbb{E}\big[\big(X - \mathbb{E}[X]\big)^2\big] = \mathbb{E}[X^2] - \mathbb{E}[X]^2, \qquad \text{resp.} \qquad \operatorname{cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y]. \tag{1.6}$$

For a measure space (E, Σ, μ), we denote by $L^p(E,\mu)$ the *Lebesgue spaces* for $1 \leq p \leq \infty$ (with real valued functions), equipped with the norm

$$\|f\|_{L^p(E,\mu)} \coloneqq \Big(\int_E |f|^p\,\mathrm{d}\mu\Big)^{\frac{1}{p}} \qquad \text{for all } f \in L^p(E,\mu) \text{ and } 1 \leq p < \infty. \tag{1.7}$$

For $p = \infty$, we define

$$\|f\|_{L^\infty(E,\mu)} \coloneqq \operatorname*{ess\,sup}_{z\in E} |f(z)| \qquad \text{for all } f \in L^\infty(E,\mu). \tag{1.8}$$

For simplicity, if $E \subset \mathbb{R}^d$ is a Borel set and μ is the Lebesgue measure with respect to E, then we write $L^p(E) \coloneqq L^p(E,\mu)$. If $1 \leq p < \infty$, for a Borel set $U \subset \mathbb{R}^d$ with $0 < |U| < \infty$, we also define the normalized norms

$$\|f\|_{\underline{L}^p(U)} = \Big(\fint_U |f(x)|^p\,\mathrm{d}x\Big)^{\frac{1}{p}} = |U|^{-\frac{1}{p}}\,\|f\|_{L^p(U)} \qquad \text{for all } f \in L^p(U). \tag{1.9}$$

For $m \in \mathbb{Z}$, we also introduce the *triadic cubes* $\square_m = \frac{1}{2}(-3^m, 3^m)^d$. If $n < m$, we define $\mathcal{Z}_n \coloneqq 3^n\mathbb{Z}^d \cap \square_m$.

2 Examples

The simplest way to construct an example is a *random checkerboard structure*. We decompose the two dimensional space into unit cubes with corners in $\mathbb{Z}^2$. We color each cube either **white** or **black** independently at random, that is we have independent random variables $\{b_z\}_{z\in\mathbb{Z}^2}$ such that for every $z \in \mathbb{Z}^d$, we have

$$\mathbb{P}[b(z) = 1] = \mathbb{P}[b(z) = 0] = \frac{1}{2}, \tag{2.1}$$

and fix two matrices $\mathbf{a}_0, \mathbf{a}_1 \in \Big\{A \in \mathbb{R}^{2\times 2}_{\text{sym}} \colon |\xi|^2 \le \xi \cdot A\xi \le \Lambda|\xi|^2 \text{ for all } \xi \in \mathbb{R}^2\Big\}$. Then we could define a random field $x \longmapsto \mathbf{a}(x)$ by

$$\mathbf{a}(x) = \mathbf{a}_{b(z)} \qquad \text{for all } x \in z = \Big[-\frac{1}{2}, \frac{1}{2}\Big)^2, \text{ for all } z \in \mathbb{Z}^2. \tag{2.2}$$

This field is in Ω and the measure $\mathbb{P}$ would be admissable.

More generally, we could consider examples using the homogeneous Poisson point process. Recall, that a homogeneous Poisson point process on the euclidean space $\mathbb{R}^d$ is an at most countable random subset $\Pi \subset \mathbb{R}^d$ with the following two properties:

- For all Borel sets $A \subset \mathbb{R}^d$, the number of points $N(A) \coloneqq \sharp(\Pi \cap A)$ follows a Poisson law of mean $|A|$.
- For every finite collection of pairwise disjoint Borel sets $A_1, \dots, A_k \subset \mathbb{R}^d$, the random variables $\{N(A_i)\}_i$ are independent.

In other words, we have a Poisson point process, if for disjoint and bounded Borel sets $A_1, \dots, A_k \subset \mathbb{R}^d$ the following holds:

$$\mathbb{P}[\sharp(\Pi \cap A_i) = j_i] = \prod_{i=1}^{k} e^{-A_i|} \frac{|A_i|^{j_i}}{j_i!}. \tag{2.3}$$

So, if we have a Poisson point process Π on $\mathbb{R}^d$, we can fix two matrices $\mathbf{a}_0$ and $\mathbf{a}_1$ as before and define a random field $x \longmapsto \mathbf{a}(x)$ by

$$\mathbf{a}(x) \coloneqq \begin{cases} \mathbf{a}_0 & \text{if } \operatorname{dist}(x, \Pi) \le \frac{1}{2}, \\ \mathbf{a}_1 & \text{otherwise.} \end{cases} \tag{2.4}$$

As above, this field is an element of Ω and with $\mathbb{P}$ as above, this measure is clearly admissable by definition of the Poisson point process. For more information, see [King].

3 The energy and sub-additive quantity ν

For the rest of this talk, we fix an open and bounded subset $U \subset \mathbb{R}^d$ with Lipschitz boundary and an element $p \in \mathbb{R}^d$. We focus on the equation

$$\begin{cases} -\operatorname{div}(\mathbf{a}\nabla u) &= 0 & \text{in } U, \\ u &= \ell_p & \text{on } \partial U, \end{cases} \tag{3.1}$$

where $\mathbf{a} \in \Omega$. We define the minimal normalized energy of this equation by $\nu = \nu(U, p)$, i.e.

$$\nu(U, p) \coloneqq \inf_{v \in \ell_p + H^1_0(U)} \frac{1}{2} \fint_U \nabla v \cdot \mathbf{a}\nabla v. \tag{3.2}$$

Since the integrand is a convex function, from the direct method of the calculus of variations, we get the existence of a minimizer $v = v(\,\cdot\,, U, p) \in \ell_p + H^1_0(U)$ of this intergral, that is

$$\nu(U, p) = \frac{1}{2} \fint_U \nabla v \cdot \mathbf{a}\nabla v. \tag{3.3}$$

We encouter some basic properties of this minimizer and the quantity ν.

Lemma 3.1 (Basic properties)
Let ν and v be defined as above. Then we have

a) **Uniform convexity**: For every $p_1, p_2 \in \mathbb{R}^d$, we have

$$\frac{1}{8}|p_1 - p_2| \leq \frac{1}{2}\nu(U, p_1) + \frac{1}{2}\nu(U, p_2) - \nu\Big(U, \frac{1}{2}p_1 + \frac{1}{2}p_2\Big) \leq \frac{\Lambda}{8}|p_1 - p_2|. \tag{3.4}$$

b) **Nomalized sub-additivity**: Let $U_1, \dots U_N$ be a finite covering of U, such that every U_i is a bounded Lipschitz domain, $U_i \subset U$ and

$$\Big|U \setminus \bigcup_{i=1}^{N} U_i\Big| = 0, \tag{3.5}$$

then we have

$$\nu(U, p) \leq \sum_{i=1}^{N} \frac{|U_i|}{|U|}\nu(U_i, p). \tag{3.6}$$

c) **First variation:** v is the unique solution of the equation

$$\begin{cases} -\mathrm{div}(\mathbf{a}\nabla u) &= 0 \qquad \text{in } U, \\ u &= \ell_p \qquad \text{on } \partial U. \end{cases} \tag{3.7}$$

In other words, v is the unique minimizer of the energy ν if and only if

$$v \in \ell_p + H_0^1(U) \qquad \text{and} \qquad \int \nabla w \cdot \mathbf{a}\nabla v = 0 \qquad \text{for all } w \in H_0^1(U). \tag{3.8}$$

d) **Quadratic response**: For every $w \in H_0^1(U)$, we have

$$\frac{1}{2}\fint_U |\nabla w - \nabla v|^2 \leq \frac{1}{2}\fint_U \nabla w \cdot \mathbf{a}\nabla w - \nu(U.p) \leq \frac{\Lambda}{2}\fint_U |\nabla w - \nabla v|^2. \tag{3.9}$$

The proof can be found in [ArKuMo, Lemma 1.1; p.6]. From the proof of this Lemma and from quadratic response, we get (with $v(U_i) = v(\,\cdot\,, U_i, p)$ for simplicity)

$$\frac{1}{2}\sum_{i=1}^{N}\frac{|U_i|}{|U|}\fint_U |\nabla v(U_i) - \nabla v(U)|^2 \leq \frac{1}{2}\sum_{i=1}^{N}\frac{|U_i|}{|U|}\nu(U_i, p) - \nu(U, p) \leq \frac{\Lambda}{2}\sum_{i=1}^{N}\frac{|U_i|}{|U|}\fint_U |\nabla v(U_i) - \nabla v(U)|^2. \tag{3.10}$$

In other words, the strictness of the subadditivity inequality is proportional to the weighted average of the L^2-differences of $v(U)$ and $v(U_i)$.

4 Convergence of ν

We want to study the convergence of the energy $\nu(U, p)$ when U gets bigger. For this purpose, it is convenient to work with triadic cubes. Since triadic cubes can be partitioned in smaller triadic cubes, and because of the stationarity of $\mathbb{P}$, we can see, that

$$\mathbb{E}[\nu(\square_{m+1}, p)] \leq \mathbb{E}[\nu(\square_m, p)] \qquad \text{for all } m \in \mathbb{N}. \tag{4.1}$$

In the proof of Lemma 3.1, we have seen, that

$$\frac{1}{2}|p|^2 \leq \nu(U, p) \leq \frac{\Lambda}{2}|p|^2. \tag{4.2}$$

Thus, the sequence $m \longmapsto \mathbb{E}[\nu(\square_m, p)]$ is bounded from below and we can find a limit

$$\bar{\nu}(p) := \inf_{m \in \mathbb{N}} \mathbb{E}[\nu(\square_m, p)]. \tag{4.3}$$

It is clear, that $p \longmapsto \bar{\nu}(p)$ is a quadratic form and that (4.2) also holds for $\bar{\nu}$. This allows us to determine homogenized coefficents.

Definition 4.1 (Homogenized coefficients)
Let $\bar{\nu}$ be defined as above. Then we defined the *homogenized coefficients* $\bar{\mathbf{a}} \in \mathbb{R}^{d\times d}_{\text{sym}}$ by the unique matrix, that satisfies

$$\bar{\nu}(p) = \frac{1}{2}\, p \cdot \bar{\mathbf{a}} p \qquad \text{for all } p \in \mathbb{R}^d. \tag{4.4}$$

It is clear, that $\mathbb{1}_{d\times d} \leq \bar{\mathbf{a}} \leq \Lambda \mathbb{1}_{d\times d}$. Thus, $\bar{\mathbf{a}}$ is a positive definite matrix.

Next, we want to show that we have $L^1(\Omega, \mathbb{P})$-convergence for the quantity ν, that is

$$\mathbb{E}[|\nu(\square_m, p) - \bar{\nu}(p)|] \longrightarrow 0 \qquad \text{as } m \to \infty. \tag{4.5}$$

We want to extract as much quantitative information as we can, so we introduce another quantity which the convergence rate will depend on. We define

$$\omega(m) := \sup_{p \in B_1} \mathbb{E}[\nu(\square_m, p)] - \bar{\nu}(p) \qquad \text{for all } m \in \mathbb{N}. \tag{4.6}$$

This quantity converges to zero as $m \to \infty$. This can be seen from the fact, that $p \longmapsto \mathbb{E}[\nu(\square_m, p)] - \bar{\nu}(p)$ is a quadratic form with a positive definite matrix, so

$$\omega(m) \leq \sum_{i=1}^{d} \mathbb{E}[\nu(\square_m, \mathbf{e}_i)] - \bar{\nu}(\mathbf{e}_i) \to 0, \tag{4.7}$$

with the standard basis $\{\mathbf{e}_i\}_i$ of $\mathbb{R}^d$.

Proposition 4.2 (Convergence in expectation)
With the above definitions, there exists a constant $C = C(d, \Lambda) > 0$, such that for every $p \in B_1$ and every $min \mathbb{N}$

$$\mathbb{E}[|\nu(\square_m, p) - \bar{\nu}(p)|] \leq C\, 3^{-\frac{d}{4}m} + C\, \omega\Big(\Big\lceil \frac{m}{2} \Big\rceil\Big). \tag{4.8}$$

The proof can be found in [ArKuMo, Proposition 1.4; p.12]. At least, we show that the minimizers $v = v(\,\cdot\,, \square_m, p)$ converge in an appropriate sense to ℓ_p.

Theorem 4.3 (Normalized L^2 convergence)
With the above defintions, there exists a constant $C = C(\Lambda, d)$, such that for every $p \in B_1$ and every $m \in \mathbb{N}$

$$\mathbb{E}\Big[3^{-2m}\, \big\| v(\,\cdot\,, \square_m, p) - \ell_p \big\|^2_{\underline{L}^2(\square_m)}\Big] \leq C\, 3^{-\frac{m}{4}} + C\, \omega\Big(\Big\lceil \frac{m}{4} \Big\rceil\Big). \tag{4.9}$$

The proof can be found in [ArKuMo, Theorem 1.5; p.14].

References

[ArKuMo] S. Armstrong, T. Kuusi, J.-C. Mourrat; *Quantitative stochastic homogenization and large-scale regularity* (2017).

[King] J. F. Kingman; *Poisson processes* (2002).