From Convex Integration to Flat Tori

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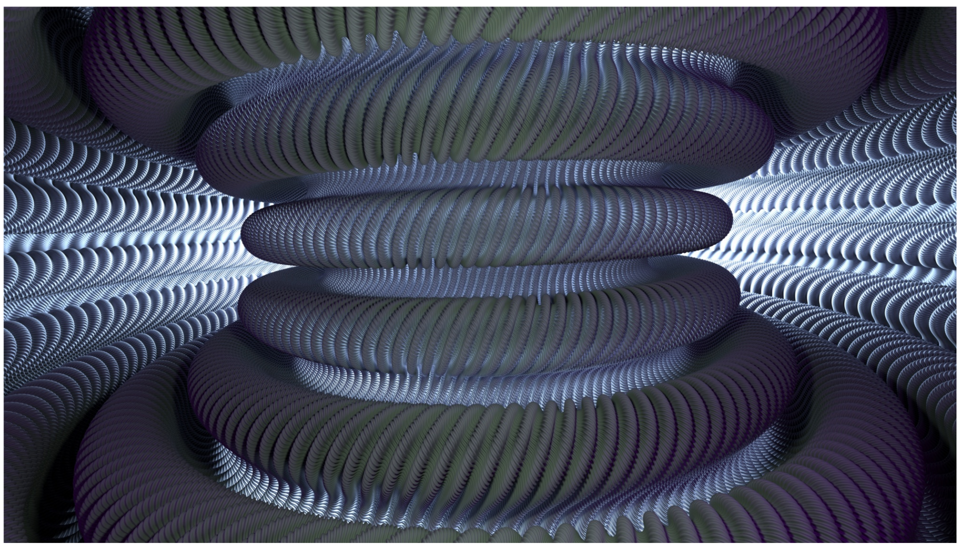
These lectures are devoted to a general method for solving differential relations: the Gromov Convex Integration Theory. We explore its simple version for first order differential relations with special attention to its geometrical and analytical foundations (Lecture 1). We introduce the notion of h-principle and then prove by using Convex Integrations that the h-principle holds for ample and open relations. We derive from this instance of h-principle many historical results in immersion-theoretic topology including the Smale eversion of the 2-sphere (Lecture 2). One special feature of Convex Integration is that it applies to solve certain classes of *closed* relations. The most interesting case is the one of the isometric relation with, as corollary, the celebrated Nash-Kuiper theorem on C^1 isometric embeddings. We show that the parametric h-principle holds for the isometric relation and, as a consequence, that the eversion of the 2-sphere can be realized by a regular homotopy of C^1 isometric immersions (Lecture 3).

One stunning corollary of the Nash-Kuiper theorem is the existence of C^1 isometric embeddings of the flat torus in the three dimensional Euclidean space. A flat torus $\mathbb{E}^2/\Lambda$ is a quotient of the Euclidean 2-plane by a lattice $\Lambda = \mathbb{Z}e_1 + \mathbb{Z}e_2$, (e_1, e_2) being a basis of $\mathbb{E}^2$. Obviously, the Gaussian curvature of a flat torus is identically zero. A classical argument shows that any C^2 complete compact surface in $\mathbb{E}^3$ has a point with positive Gaussian curvature. From the Theorema Egregium it ensues that there is no C^2 isometric embedding of any flat torus. It should be stressed that a C^1 isometric map which is not C^2 has no defined Gaussian curvature but has a defined Gauss map: the image of any C^1 isometric embedding admits a tangent space at any of its points. The unusual regularity of the Nash-Kuiper embeddings puzzles the imagination. Although the Nash-Kuiper proof is constructive, the geometry of these embeddings could barely be conceived from their original papers.

In Lecture 4, we provide an explicit construction of a C^1 isometric embedding of a flat torus based on the Convex Integration Theory. We then turn this construction into a computer implementation leading us to the visualisation of an isometrically embedded flat torus. The pictures reveal a geometric object in-between fractals and ordinary surfaces.



Interior View of an Isometrically Embedded Flat Torus

References

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