Boundary value problems for the Willmore and the Helfrich functional for surfaces of revolution

Hans-Christoph Grunau

Otto-von-Guericke-Universität Magdeburg

This talk concerns joint works with A. Dall'Acqua, K. Deckelnick, M. Doemeland, S. Eichmann, and S. Okabe.

A special form of the Helfrich energy for a sufficiently smooth (two dimensional) surface $S \subset \mathbb{R}^3$ (with or without boundary) is defined by

$$\mathscr{H}_\varepsilon(S) := \int_S H^2\,dS + \varepsilon \int_S dS,$$

where H denotes the mean curvature of S. The first integral may be considered as a bending energy and the second as surface (stretching) energy. $\mathscr{W}(S) := \mathscr{H}_0(S)$ is called the Willmore functional. We consider surfaces of revolution S

$$(x,\varphi) \mapsto \big(x, u(x)\cos\varphi, u(x)\sin\varphi\big), \quad x \in [-1,1],\ \varphi \in [0,2\pi],$$

with smooth strictly positive profile curve u subject to Dirichlet boundary conditions

$$u(-1) = \alpha, \quad u(1) = \beta, \quad u'(\pm 1) = 0$$

and aim at minimising $\mathscr{H}_\varepsilon$. Thanks to these boundary conditions the Gauss curvature integral $\int_S K\,dS$ becomes a constant and needs not to be considered.

In the first part of the lecture I shall consider the Willmore case, i.e. $\varepsilon = 0$. After briefly recalling minimisation in the symmetric case $\alpha = \beta$ (see [1,4]) I shall show how much more complicated the problem becomes for $\alpha \neq \beta$. Only when α and β do not differ too much, the profile curve will remain a graph while in general it will become a nonprojectable curve, see [3].

In the second part, $\mathscr{H}_\varepsilon$ is considered for $\varepsilon \in [0,\infty)$, but again in the symmetric setting $\alpha = \beta$. For $\alpha \geq \alpha_m = c_m \cosh(\frac{1}{c_m}) \approx 1.895$ with $c_m \approx 1.564$ the unique solution of the equation $\frac{2}{c} = 1 + e^{-2/c}$, when one has a catenoid v_α which globally minimises the surface energy, we find minimisers u_ε for any $\varepsilon \geq 0$ and show uniform and locally smooth convergence $u_\varepsilon \to v_\alpha$ under the singular limit $\varepsilon \to \infty$. These results are collected in [2].

At the end I shall briefly mention recent work on obstacle problems [5].

[1] A. Dall'Acqua, K. Deckelnick, and H.-Ch. Grunau, Classical solutions to the Dirichlet problem for Willmore surfaces of revolution, *Adv. Calc. Var.* **1** (2008), 379-397.

[2] K. Deckelnick, H.-Ch. Grunau, and M. Doemeland, Boundary value problems for the Helfrich functional for surfaces of revolution - Existence and asymptotic behaviour, Calc. Var. Partial Differ. Equ. **60** (2021), Article number 32.

[3] S. Eichmann and H.-Ch. Grunau, Existence for Willmore surfaces of revolution satisfying non-symmetric Dirichlet boundary conditions, *Adv. Calc. Var.* **12** (2019), 333–361.

[4] H.-Ch. Grunau, The asymptotic shape of a boundary layer of symmetric Willmore surfaces of revolution. In: C. Bandle et al. (eds.), Inequalities and Applications 2010. *International Series of Numerical Mathematics* **161** (2012), 19-29.

[5] H.-Ch. Grunau and S. Okabe, Willmore obstacle problems under Dirichlet boundary conditions, submitted.