Generalized Rational Zeta Series with Two Parameters

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It can be shown that any real number can be written as a rational zeta series. Examples:

$$\sum_{n=2}^{\infty} [\zeta(n) - 1] = 1, \quad \sum_{n=2}^{\infty} \frac{\zeta(n) - 1}{n} = 1 - \gamma$$

where $\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log(n) \right)$ is the Euler-Mascheroni constant.

Definition.

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In 1734, Leonhard Euler showed $\zeta(2) = \pi^2/6$. More generally,

$$\zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k+1} B_{2k} (2\pi)^{2k}}{2(2k)!}, \quad k \in \mathbb{N}_0,$$

where

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n, \quad |z| < 2\pi.$$

One of many appearances:

$$\cot(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} B_{2n}}{(2n)!} x^{2n-1} = -2 \sum_{n=0}^{\infty} \frac{\zeta(2n)}{\pi^{2n}} x^{2n-1}, \quad |x| < \pi. \tag{2}$$

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Note that $\beta(2) = G$ is known as Catalan's constant.

The Clausen function (see [3], [4], [7], [9], [11], [12], [14]) or Clausen's integral is defined as

$$\mathrm{Cl}_2(\theta) := \sum_{k=1}^{\infty} \frac{\sin(k\theta)}{k^2} = -\int_0^{\theta} \log\left(2\sin\left(\frac{\phi}{2}\right)\right) d\phi. \tag{4}$$

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Its Taylor series is given by

$$\frac{\mathrm{Cl}_2(\theta)}{\theta} = 1 - \log|\theta| + \sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(2n+1)}\left(\frac{\theta}{2\pi}\right)^{2n}, \quad |\theta| < 2\pi. \tag{5}$$

Higher order Clausen-type functions:

$$\mathrm{Cl}_{2m}(\theta) := \sum_{k=1}^{\infty} \frac{\sin(k\theta)}{k^{2m}}, \quad \mathrm{Cl}_{2m+1}(\theta) := \sum_{k=1}^{\infty} \frac{\cos(k\theta)}{k^{2m+1}}. \qquad (6)$$

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Using (1) and (3), we have

$$\mathrm{Cl}_{2m}(\pi) = 0, \quad \mathrm{Cl}_{2m+1}(\pi) = -\frac{(4^m - 1)\zeta(2m+1)}{4^m}, \tag{7}$$

and

$$\mathrm{Cl}_{2m}(\pi/2) = \beta(2m), \quad \mathrm{Cl}_{2m+1}(\pi/2) = -\frac{(4^m - 1)\zeta(2m+1)}{2^{4m+1}}. \tag{8}$$

We can also see

$$\frac{d}{d\theta}\,\mathrm{Cl}_{2m}(\theta) = \mathrm{Cl}_{2m-1}(\theta), \quad \frac{d}{d\theta}\,\mathrm{Cl}_{2m+1}(\theta) = -\,\mathrm{Cl}_{2m}(\theta), \qquad (9)$$

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$$\int_0^\theta \mathrm{Cl}_{2m}(x)\,dx = \zeta(2m+1) - \mathrm{Cl}_{2m+1}(\theta), \quad \int_0^\theta \mathrm{Cl}_{2m-1}(x)\,dx = \mathrm{Cl}_{2m}(\theta). \tag{10}$$

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Note

$$\operatorname{Cl}_1(\theta) = -\log\left(2\sin\left(\frac{\theta}{2}\right)\right), \quad |\theta| < 2\pi. \tag{11}$$

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By induction, for $m \geq 3$,

$$\mathrm{Cl}_m(z) = (-1)^{\lfloor \frac{m-1}{2} \rfloor} \left(\sum_{k=1}^{\lfloor \frac{m-1}{2} \rfloor} \frac{(-1)^k z^{m-2k-1}}{(m-2k-1)!} \zeta(2k+1) + \int_0^z \frac{(z-t)^{m-3}}{(m-3)!} \mathrm{Cl}_2(t)\, dt \right). \quad (12)$$

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where $\delta_{j,k}$ is the Kronecker delta function.

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Proof. Denoting the left hand side as $f(z)$ and the right hand side as $g(z)$, one can show $f(0) = g(0)$ and $f'(z) = g'(z)$, so thus $f(z) = g(z)$.

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Using (2) we can also say

$$\int_0^{\pi z} x^p \cot(x)\, dx = -2 \int_0^{\pi z} \sum_{n=0}^{\infty} \frac{\zeta(2n)}{\pi^{2n}} x^{2n-1+p}\, dx$$

$$= -2(\pi z)^p \sum_{n=0}^{\infty} \frac{\zeta(2n) z^{2n}}{2n+p},$$

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$$= -2(\pi z)^p\sum_{n=0}^{\infty}\frac{\zeta(2n)z^{2n}}{2n+p},$$

and with (13),

$$-2\sum_{n=0}^{\infty}\frac{\zeta(2n)z^{2n}}{2n+p} = \sum_{k=0}^{p}\frac{p!(-1)^{\lfloor\frac{k+3}{2}\rfloor}}{(p-k)!(2\pi z)^k}\mathrm{Cl}_{k+1}(2\pi z)$$
$$+\,\delta_{\lfloor\frac{p}{2}\rfloor,\frac{p}{2}}\frac{p!(-1)^{\frac{p}{2}}}{(2\pi z)^p}\zeta(p+1). \quad (14)$$

For $z = 1/2$ and $z = 1/4$,

$$-2\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+p)4^n} = \log 2 + \sum_{k=1}^{\lfloor \frac{p}{2} \rfloor}\frac{p!(-1)^k(4^k-1)\zeta(2k+1)}{(p-2k)!(2\pi)^{2k}} + \delta_{\lfloor \frac{p}{2} \rfloor, \frac{p}{2}}\frac{p!(-1)^{\frac{p}{2}}\zeta(p+1)}{\pi^p}, \quad (15)$$

$$-2\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+p)16^n} = \frac{1}{2}\log 2 + \frac{1}{2}\sum_{k=1}^{\lfloor \frac{p}{2} \rfloor}\frac{p!(-1)^k(4^k-1)\zeta(2k+1)}{(p-2k)!(2\pi)^{2k}} - \frac{\pi}{2}\sum_{k=1}^{\lfloor \frac{p+1}{2} \rfloor}\frac{p!(-4)^k\beta(2k)}{(p+1-2k)!\pi^{2k}} + \delta_{\lfloor \frac{p}{2} \rfloor, \frac{p}{2}}\frac{p!(-1)^{\frac{p}{2}}2^p\zeta(p+1)}{\pi^p}. \quad (16)$$

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$$\sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n+1)4^n} = -\frac{1}{2}\log 2,$$

and

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Subtracting amd rearranging, we have a nice formula for G:

$$G = \frac{\pi}{2}\sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n+1)4^n}\left(1 - \frac{2}{4^n}\right)$$

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$$\sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n+3)16^n} = -\frac{1}{4}\log 2 + \frac{9\zeta(3)}{8\pi^2} - \frac{3G}{\pi} + \frac{24\beta(4)}{\pi^3}.$$

Theorem 2.

$$\int_0^{2\pi z} x^p \,\mathrm{Cl}_m(x)\, dx = - \sum_{k=m+1}^{m+p+1} \frac{(2\pi z)^{p+m+1-k} p!(-1)^{\lfloor \frac{m}{2} \rfloor}(-1)^{\lfloor \frac{k}{2} \rfloor}}{(p+m+1-k)!} \mathrm{Cl}_k(2\pi z) + \delta_{\lfloor \frac{p+m}{2} \rfloor, \frac{p+m}{2}} (-1)^{\lfloor \frac{m}{2} \rfloor} p!(-1)^{\frac{p+m}{2}} \zeta(p+m+1), \;\; p \in \mathbb{N}_0, \;\; m \in \mathbb{N}. \tag{17}$$

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Proof. Same as the proof for Theorem 1.

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Proof. Same as the proof for Theorem 1. □

Using (5) and (12),

$$\int_0^{2\pi z} x^p \,\mathrm{Cl}_m(x)\, dx = (-1)^{\lfloor \frac{m-1}{2} \rfloor} \int_0^{2\pi z} x^p *$$

$$\left(\sum_{k=1}^{\lfloor \frac{m-1}{2} \rfloor} \frac{(-1)^k x^{m-2k-1}}{(m-2k-1)!} \zeta(2k+1) + \int_0^x \frac{(x-t)^{m-3}}{(m-3)!} \mathrm{Cl}_2(t)\, dt \right) dx$$

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$$= \sum_{k=1}^{\lfloor \frac{m-1}{2} \rfloor} \frac{(-1)^{\lfloor \frac{m-1}{2} \rfloor + k} (2\pi z)^{m+p-2k}}{(m-1-2k)!(m+p-2k)} \zeta(2k+1) + \frac{(-1)^{\lfloor \frac{m-1}{2} \rfloor}}{(m-3)!} *$$

$$\int_0^{2\pi z} \int_0^x x^p (x-t)^{m-3} \left(t - t \log t + \sum_{n=1}^{\infty} \frac{\zeta(2n) t^{2n+1}}{n(2n+1)(2\pi)^{2n}} \right) dt\, dx$$

Evaluating the first integral,

$$\int_0^{2\pi z} x^p \,\mathrm{Cl}_m(x)\, dx = \sum_{k=1}^{\lfloor \frac{m-1}{2} \rfloor} \frac{(-1)^{\lfloor \frac{m-1}{2} \rfloor + k}(2\pi z)^{m+p-2k}}{(m-1-2k)!(m+p-2k)}\zeta(2k+1)$$
$$+ \frac{(-1)^{\lfloor \frac{m-1}{2} \rfloor}}{(m-3)!} \int_0^{2\pi z} x^p \left(\frac{x^{m-1}(H_{m-1} - \log x)}{(m-1)(m-2)} \right.$$
$$\left. + \sum_{n=1}^{\infty} \frac{2\zeta(2n)\Gamma(m-2)x^{m+2n-1}}{2n(2n+1)(2n+2)\ldots(2n+m-1)(2\pi)^{2n}} \right) dx,$$

H_n is the n-th harmonic number, $H_0 = 0$.

And the second integral...

$$\int_0^{2\pi z} x^p \,\mathrm{Cl}_m(x)\, dx = \frac{(2\pi z)^{p+m}(-1)^{\lfloor \frac{m-1}{2} \rfloor}}{(m-1)!}\Bigg(\frac{H_{m-1} - \log(2\pi z)}{(p+m)} + \frac{1}{(p+m)^2} + \sum_{k=1}^{\lfloor \frac{m-1}{2} \rfloor} \frac{(m-1)!(-1)^k \zeta(2k+1)}{(m-1-2k)!(p+m-2k)(2\pi z)^{2k}} + \sum_{n=1}^{\infty} \frac{(m-1)!\zeta(2n) z^{2n}}{n(2n+1)\dots(2n+m-1)(2n+p+m)}\Bigg). \quad (18)$$

Using (17) and rearranging, we find

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)z^{2n}}{n(2n+1)\dots(2n+m-1)(2n+m+p)} = (-1)^m p! *$$
$$\left(\sum_{k=m}^{m+p} \frac{(-1)^{\lfloor \frac{k+1}{2} \rfloor} \mathrm{Cl}_{k+1}(2\pi z)}{(p+m-k)!(2\pi z)^k} - \delta_{\lfloor \frac{p+m}{2} \rfloor, \frac{p+m}{2}} \frac{(-1)^{\frac{p+m}{2}}}{(2\pi z)^{p+m}} \zeta(p+m+1) \right)$$
$$+ \frac{\log(2\pi z) - H_{m-1}}{(m-1)!(p+m)} - \sum_{k=1}^{\lfloor \frac{m-1}{2} \rfloor} \frac{(-1)^k \zeta(2k+1)}{(m-1-2k)!(m+p-2k)(2\pi z)^{2k}}$$
$$- \frac{1}{(m-1)!(p+m)^2}. \quad (19)$$

For $z = 1/2$,

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(2n+1)\dots(2n+m-1)(2n+m+p)4^n} = (-1)^{m+1} p! *$$

$$\left(\sum_{k=\lfloor \frac{m+1}{2} \rfloor}^{\lfloor \frac{p+m}{2} \rfloor} \frac{(-1)^k (4^k - 1)\zeta(2k+1)}{(p+m-2k)!(2\pi)^{2k}} + \delta_{\lfloor \frac{p+m}{2} \rfloor, \frac{p+m}{2}} \frac{(-1)^{\frac{p+m}{2}}}{\pi^{p+m}} \zeta(p+m+1) \right)$$

$$+ \frac{\log \pi - H_{m-1}}{(m-1)!(p+m)} - \sum_{k=1}^{\lfloor \frac{m-1}{2} \rfloor} \frac{(-1)^k \zeta(2k+1)}{(m-1-2k)!(p+m-2k)\pi^{2k}}$$

$$- \frac{1}{(m-1)!(p+m)^2}, \quad (20)$$

and for $z = 1/4$,

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(2n+1)\dots(2n+m-1)(2n+m+p)16^n} = \frac{(-1)^m p!}{2} *$$
$$\left(\sum_{k=\lfloor \frac{m+2}{2} \rfloor}^{\lfloor \frac{p+m+1}{2} \rfloor} \frac{\pi(-4)^k \beta(2k)}{(p+m+1-2k)!\pi^{2k}} - \sum_{k=\lfloor \frac{m+1}{2} \rfloor}^{\lfloor \frac{p+m}{2} \rfloor} \frac{(-1)^k (4^k-1)\zeta(2k+1)}{(p+m-2k)!(2\pi)^{2k}} \right.$$
$$\left. - \delta_{\lfloor \frac{p+m}{2} \rfloor, \frac{p+m}{2}} \frac{(-1)^{\frac{p+m}{2}} 2^{p+m+1}}{\pi^{p+m}} \zeta(p+m+1) \right) + \frac{\log(\pi/2) - H_{m-1}}{(m-1)!(p+m)}$$
$$- \sum_{k=1}^{\lfloor \frac{m-1}{2} \rfloor} \frac{(-4)^k \zeta(2k+1)}{(m-1-2k)!(p+m-2k)\pi^{2k}} - \frac{1}{(m-1)!(p+m)^2}. \tag{21}$$

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and

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(2n+1)16^n} = \frac{2G}{\pi} - 1 + \log\left(\frac{\pi}{2}\right),$$

both of which are famous series ([13], [11]).

Remark 2.

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$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(2n+1)(2n+3)(n+2)16^n}$$

converges very closely to 0.0023 (0.00229999499895073...). Using (21) a few times for different m and p, one can show that

$$\zeta(5) = \frac{4\pi^4}{1581}\left(\frac{19}{12} - \log\left(\frac{\pi}{2}\right) + \frac{96}{\pi^3}\beta(4) + 6\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(2n+1)(2n+3)(n+2)16^n}\right),$$

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$$\zeta(5) \approx \frac{4\pi^4}{1581}\left(\frac{19}{12} - \log\left(\frac{\pi}{2}\right) + \frac{96}{\pi^3}\beta(4) + 6*0.0023\right) = 1.03692775588\ldots$$

which has a small error of $7.395 * 10^{-10}$.

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Using the binomial theorem,

$$\int_0^{\pi z}\int_0^x x^p(x-t)^m t\cot(t)\,dt\,dx = \sum_{j=0}^{m}(-1)^j\binom{m}{j}\int_0^{\pi z} x^{p+m-j}\int_0^x t^{j+1}\cot(t)\,dt\,dx$$

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and now we can use theorem 1.

$$\int_0^{\pi z}\int_0^{x} x^p (x-t)^m t\cot(t)\,dt\,dx =$$

$$\sum_{j=0}^{m}\sum_{k=0}^{j+1}\binom{m}{j}\frac{(-1)^j (j+1)!(-1)^{\lfloor\frac{k+3}{2}\rfloor}}{(j+1-k)!2^k}\int_0^{\pi z} x^{p+m+1-k}\,\mathrm{Cl}_{k+1}(2x)\,dx$$

$$+\sum_{j=0}^{m}\delta_{\lfloor\frac{j+1}{2}\rfloor,\frac{j+1}{2}}\frac{(-1)^j (j+1)!(-1)^{\frac{j+1}{2}}}{2^{j+1}}\binom{m}{j}\zeta(j+2)\int_0^{\pi z} x^{p+m-j}\,dx$$

$$=\sum_{j=0}^{m}\sum_{k=0}^{j+1}\binom{m}{j}\frac{(-1)^j (j+1)!(-1)^{\lfloor\frac{k+3}{2}\rfloor}}{(j+1-k)!2^{p+m+2}}\int_0^{2\pi z} u^{p+m+1-k}\,\mathrm{Cl}_{k+1}(u)\,du$$

$$-\sum_{k=1}^{\lfloor\frac{m+1}{2}\rfloor}\frac{(2k)(-1)^k m!(\pi z)^{p+m+2-2k}}{2^{2k}(m+1-2k)!(p+m+2-2k)}\zeta(2k+1)$$

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$$\int_0^{\pi z}\int_0^{x} x^p (x-t)^m t\cot(t)\, dt\, dx = -\frac{(-1)^{\lfloor \frac{m}{2}\rfloor} m!}{2^{p+m+2}} *$$

$$\int_0^{2\pi z} u^{p+1}\,\mathrm{Cl}_{m+1}(u)\, du + \frac{(m+1)!(-1)^{\lfloor \frac{m+1}{2}\rfloor}}{2^{p+m+2}} \int_0^{2\pi z} u^{p}\,\mathrm{Cl}_{m+2}(u)\, du$$

$$+\sum_{k=1}^{\lfloor \frac{m+1}{2}\rfloor} \frac{(2k)(-1)^{k+1} m!(\pi z)^{p+m+2-2k}\zeta(2k+1)}{2^{2k}(m+1-2k)!(p+m+2-2k)}.$$

Evaluating the sums on j and k,

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Now we can integrate the first integral by parts and use theorem 2...

Rearranging and simplifying, we will arrive at

$$\int_0^{\pi z}\int_0^x x^p(x-t)^m t\cot(t)\,dt\,dx = (p+m+2)(-1)^m p!m! *$$
$$\left(2\pi z\sum_{k=m+3}^{p+m+3}\frac{(-1)^{\lfloor\frac{k}{2}\rfloor}(\pi z)^{p+m+2}\,\mathrm{Cl}_k(2\pi z)}{(p+m+3-k)!(2\pi z)^k}+\delta_{\lfloor\frac{p+m}{2}\rfloor,\frac{p+m}{2}} *\right.$$
$$\left.\frac{(-1)^{\frac{p+m}{2}}\zeta(p+m+3)}{2^{m+p+2}}\right)-\frac{(-1)^{\lfloor\frac{m+1}{2}\rfloor}m!(\pi z)^{p+1}\,\mathrm{Cl}_{m+2}(2\pi z)}{2^{m+1}}$$
$$+\sum_{k=1}^{\lfloor\frac{m+1}{2}\rfloor}\frac{(2k)(-1)^{k+1}m!(\pi z)^{p+m+2}\zeta(2k+1)}{(2\pi z)^{2k}(m+1-2k)!(p+m+2-2k)}. \tag{22}$$

Another way to evaluate this double integral is to use the power series for $\cot(x)$. Doing so, we see

$$\int_0^{\pi z}\int_0^x x^p(x-t)^m t\cot(t)\,dt\,dx$$

$$=-2\int_0^{\pi z} x^p\sum_{n=0}^{\infty}\frac{\zeta(2n)}{\pi^{2n}}\int_0^x (x-t)^m t^{2n}\,dt\,dx$$

$$=-2\sum_{n=0}^{\infty}\frac{\zeta(2n)m!\Gamma(2n+1)}{\pi^{2n}\Gamma(m+2n+2)}\int_0^{\pi z} x^{2n+m+p+1}\,dx$$

$$=-2\sum_{n=0}^{\infty}\frac{\zeta(2n)m!(\pi z)^{m+p+2}z^{2n}}{(2n+1)\ldots(2n+m+1)(2n+m+p+2)}.$$

Putting this and (22) together, we find

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$$\sum_{n=0}^{\infty} \frac{\zeta(2n)z^{2n}}{(2n+1)\ldots(2n+m+1)(2n+m+p+2)} = (p+m+2)*$$
$$(-1)^{m+1}p!\left(\sum_{k=m+3}^{p+m+3} \frac{\pi z(-1)^{\lfloor \frac{k}{2} \rfloor}\,\mathrm{Cl}_k(2\pi z)}{(p+m+3-k)!(2\pi z)^k} + \delta_{\lfloor \frac{p+m}{2} \rfloor, \frac{p+m}{2}} *\right.$$
$$\left.\frac{(-1)^{\frac{p+m}{2}}\zeta(p+m+3)}{2(2\pi z)^{m+p+2}}\right) + \frac{(-1)^{\lfloor \frac{m+1}{2} \rfloor}\,\mathrm{Cl}_{m+2}(2\pi z)}{2(2\pi z)^{m+1}}$$
$$+\sum_{k=1}^{\lfloor \frac{m+1}{2} \rfloor} \frac{k(-1)^k\zeta(2k+1)}{(2\pi z)^{2k}(m+1-2k)!(m+p+2-2k)}. \qquad (23)$$

For $z = 1/2$,

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$$\sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n+1)\dots(2n+m+1)(2n+m+p+2)4^n} = \frac{p!(m+p+2)}{2} *$$

$$(-1)^m \left(\sum_{k=\lfloor \frac{m+3}{2} \rfloor}^{\lfloor \frac{p+m+2}{2} \rfloor} \frac{(-1)^k(4^k-1)\zeta(2k+1)}{(p+m+2-2k)!(2\pi)^{2k}} - \delta_{\lfloor \frac{p+m}{2} \rfloor, \frac{p+m}{2}} * \right.$$

$$\left. \frac{(-1)^{\frac{p+m}{2}}\zeta(p+m+3)}{\pi^{m+p+2}} \right) - \delta_{\lfloor \frac{m+1}{2} \rfloor, \frac{m+1}{2}} \frac{(-1)^{\frac{m+1}{2}}(2^{m+1}-1)\zeta(m+2)}{2(2\pi)^{m+1}}$$

$$+ \sum_{k=1}^{\lfloor \frac{m+1}{2} \rfloor} \frac{k(-1)^k\zeta(2k+1)}{\pi^{2k}(m+1-2k)!(m+p+2-2k)}, \qquad (24)$$

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$$\sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n+1)\dots(2n+m+1)(2n+m+p+2)16^n} = \frac{p!(m+p+2)}{4} *$$

$$\left(\sum_{k=\lfloor \frac{m+3}{2} \rfloor}^{\lfloor \frac{p+m+2}{2} \rfloor} \frac{(-1)^{m+k}(4^k-1)\zeta(2k+1)}{(p+m+2-2k)!(2\pi)^{2k}} - \sum_{k=\lfloor \frac{m+4}{2} \rfloor}^{\lfloor \frac{p+m+3}{2} \rfloor} \frac{\pi(-1)^m(-4)^k\beta(2k)}{(p+m+3-2k)!\pi^{2k}} \right.$$

$$\left. -\delta_{\lfloor \frac{p+m}{2} \rfloor, \frac{p+m}{2}} \frac{(-1)^{\frac{p+m}{2}}(-1)^m 2^{p+m+3}\zeta(p+m+3)}{\pi^{p+m+2}} \right) + \delta_{\lfloor \frac{m}{2} \rfloor, \frac{m}{2}} \frac{(-1)^{\lfloor \frac{m+1}{2} \rfloor}}{\pi^{m+1}} *$$

$$2^m\beta(m+2) + \sum_{k=1}^{\lfloor \frac{m+1}{2} \rfloor} \frac{k(-4)^k\zeta(2k+1)}{\pi^{2k}(m+1-2k)!(m+p+2-2k)} - \delta_{\lfloor \frac{m+1}{2} \rfloor, \frac{m+1}{2}} *$$

$$\frac{(-1)^{\frac{m+1}{2}}(2^{m+1}-1)\zeta(m+2)}{4(2\pi)^{m+1}}. \quad (25)$$

Remark.

Remark. For $m = 0$ and $p = 0$, we have

$$\sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n+1)(n+1)4^n} = -\frac{7\zeta(3)}{2\pi^2},$$

and

$$\sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n+1)(n+1)16^n} = \frac{2G}{\pi} - \frac{35\zeta(3)}{4\pi^2},$$

the first of which was rediscovered by Ewell [8].

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Using the digamma function and negapolygammas, we can have $\zeta(2n+1)$ on the numerator of these zeta series instead of $\zeta(2n)$.

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$$\psi^{(0)}(z) = \psi(z) = \frac{d}{dz}\log\Gamma(z), \quad \psi^{(-1)}(z) = \log\Gamma(z),$$

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The Taylor series for $\log\Gamma(z)$ is

$$\log\Gamma(z) = -\log z - \gamma z + \sum_{k=2}^{\infty}\frac{(-1)^k\zeta(k)}{k}z^k,\ \ |z|<1.$$

We can split the above series into odd k and even k and use the formulas for $\zeta(2n)$ to establish rational zeta series for $\zeta(2n+1)$.

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Thank You!