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PDE and Analysis Seminar
Monday, November 27, 2023 - 15:30

Sobolev and Poincaré inequalities for differential forms in Heisenberg groups.

Heisenberg groups $\mathbb{H}^n$, $n \geq 1$, are connected, simply connected Lie groups whose Lie algebra is the central extensions

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2, \quad \text{with } \mathfrak{h}_2 = \mathbb{R} = Z(\mathfrak{h}),$$

with bracket $\mathfrak{h}_1 \otimes \mathfrak{h}_1 \to \mathfrak{h}_2 = \mathbb{R}$ being a non-degenerate skew-symmetric 2-form. Due to its stratification, the Heisenberg Lie algebra admits a one parameter group of automorphisms δ_t,

$$\delta_t = t \text{ on } \mathfrak{h}_1, \quad \delta_t = t^2 \text{ on } \mathfrak{h}_2,$$

which are counterparts of the usual Euclidean dilations in $\mathbb{R}^N$. Since $\dim \mathfrak{h} = 2n+1$, through exponential coordinates, $\mathbb{H}^n$ can be identified with $\mathbb{R}^{2n+1}$ endowed with a non-commutative product given by the Baker-Campell-Hausdorff formula.

The stratification of the Lie algebra $\mathfrak{h}$ yields a lack of homogeneity of de Rham's exterior differential with respect to group dilations δ_λ. The so-called Rumin's complex $(E_0^\bullet, d_c)$ is meant precisely to bypass the lack of homogeneity of de Rham's complex through a new complex that is still homotopic to de Rham's complex. A further crucial feature of Rumin's complex is that d_c is an operator of order 1 on h-forms when $h \neq n$ and of order 2 when $h = n$. In addition, $\mathbb{H}^n$ can be endowed with an homogeneous norm that induces a structure of metric space $(\mathbb{H}^n, d_c)$. It is important to stress that the Hausdorff dimension of $(\mathbb{H}^n, d_c)$ is $Q := 2n+2$. In this talk we present a family of inequalities proved in collaboration with A. Baldi (Bologna) and P. Pansu (Orsay). Denote by $B := B(x,r)$ a generic d_c-ball.

To give a gist of our results, we present the notion of interior Poincaré inequality in $E_0^\bullet$.

Definition 0.1. *Let $B = B(e,1)$ and $B(\lambda) = B(e,\lambda)$. Given $1 \leq h \leq 2n+1$ and $1 \leq p \leq q \leq \infty$, we say that the interior $\mathbb{H}$-Poincaré$_{p,q}$ inequality holds in E_0^h if there exist constants $\lambda > 1$ and C such that, for every d_c-exact differential h-form ω in $L^p(B(\lambda); E_0^h)$ there exists a differential $(h-1)$-form ϕ in $L^q(B, E_0^{h-1})$ such that $d_c\phi = \omega$ and*

$$\|\phi\|_{L^q(B,E_0^{h-1})} \leq C\, \|\omega\|_{L^p(B(\lambda),E_0^h)} \qquad \text{interior } \mathbb{H}\text{-Poincaré}_{p,q}(h). \tag{1}$$

We can prove that

Theorem 0.2 (Poincaré inequality). *If $1 \leq h \leq 2n+1$, we have:*

i) *if $h \neq n+1, 2n+1$ and $1 \leq p < Q$, then the interior $\mathbb{H}$-Poincaré$_{p,pQ/(Q-p)}(h)$ holds;*
ii) *if $h = n+1$ and $1 \leq p < Q/2$, then the interior $\mathbb{H}$-Poincaré$_{p,pQ/(Q-2p)}(n+1)$ holds;*
iii) *if $h = 2n+1$ and $1 < p < Q$, then the interior $\mathbb{H}$-Poincaré$_{p,pQ/(Q-p)}(h)$ holds.*

The case $p = 1$ requires an approach *ad hoc*. The other endpoint result $p = N$, $q = \infty$, is more delicate. Indeed, it is well known that the interior Poincaré$_{N,\infty}(1)$ fails to hold in

$\mathbb{R}^N$, and has to be replaced by the so-called Trudinger exponential estimate or by the more precise Adams-Trudinger inequality.

However, rather surprisingly, in 2007 Bourgain & Brezis proved that a global Poincaré$_{N,\infty}(h)$ holds for $1 < h < N-1$. An analogous result can be proved in $\mathbb{H}^n$ provided we replace N by Q or $Q/2$ according to the degree of the form.