

**“CHARACTERIZING GEOMETRIC AND TOPOLOGICAL
PROPERTIES OF SETS AND SPACES
VIA FIXED POINT PROPERTIES.”**

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ABSTRACT. A major theme in Fixed Point Theory is characterizing geometric and topological properties of sets or spaces in terms of fixed point properties. E.g., from Brouwer (1912), Schauder (1930) and Klee (1955), we know that closed and convex subsets K of a Banach space X are norm compact if and only every continuous map on K has a fixed point. Also, Jaggi (1983) and Kassay (1986) proved that a reflexive Banach space X has normal structure if and only if every Jaggi nonexpansive map on a closed bounded convex subset of X has a fixed point. And Maurey (1980) and Dowling-L (1997) showed that for a subspace X of $(L^1[0, 1], \|\cdot\|_1)$, X is reflexive if and only if every nonexpansive map on a closed bounded convex subset of X has a fixed point. Moreover, by Maurey (1980) and Dowling-L-Turett (2004), a closed bounded convex subset K of $(c_0, \|\cdot\|_\infty)$ is weakly compact if and only if every nonexpansive map on K has a fixed point. And L-Nezir (2014), using a theorem of Domínguez Benavides (2009), proved that a Banach lattice $(X, \|\cdot\|)$ is reflexive if and only if X has an equivalent norm $\|\cdot\|^\sim$ such that every $\|\cdot\|^\sim$ -cascading nonexpansive map has fixed point. It is a fascinating open problem to discover a geometric condition on equivalent norms $\|\cdot\|$ on $(\ell^1, \|\cdot\|_1)$, characterizing when every $\|\cdot\|$ -nonexpansive map on a closed bounded convex subset has a fixed point. We will discuss the above, and recent developments related to this circle of ideas.

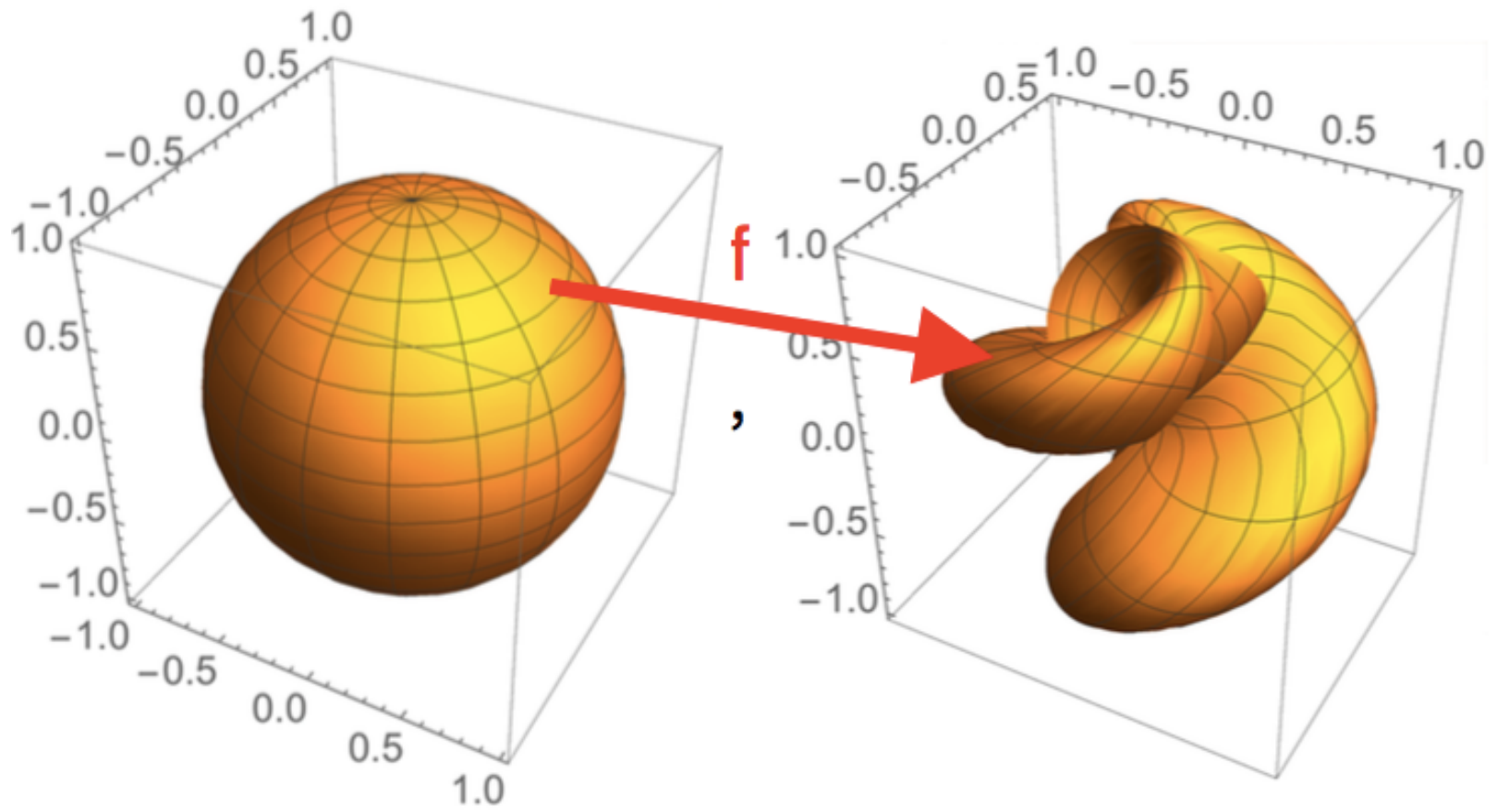
1. INTRODUCTION

One form of Brouwer's fixed point theorem says:

if B is the usual closed unit ball in \mathbb{R}^n , then every continuous map f from B into B has a fixed point.

See the example below: no matter how convoluted the range of f is, at least one point in B ends up where it started.

In 1930, Schauder extended the above theorem to an arbitrary Banach space X and an arbitrary norm compact, convex subset B of X .



In 1955, Klee considered an arbitrary closed, convex subset C of X that is not norm compact. His construction follows. (Also see, for example, Goebel (2002).) There exists a constant $a > 0$ and a sequence $(v_n)_{n \geq 0}$ in C satisfying

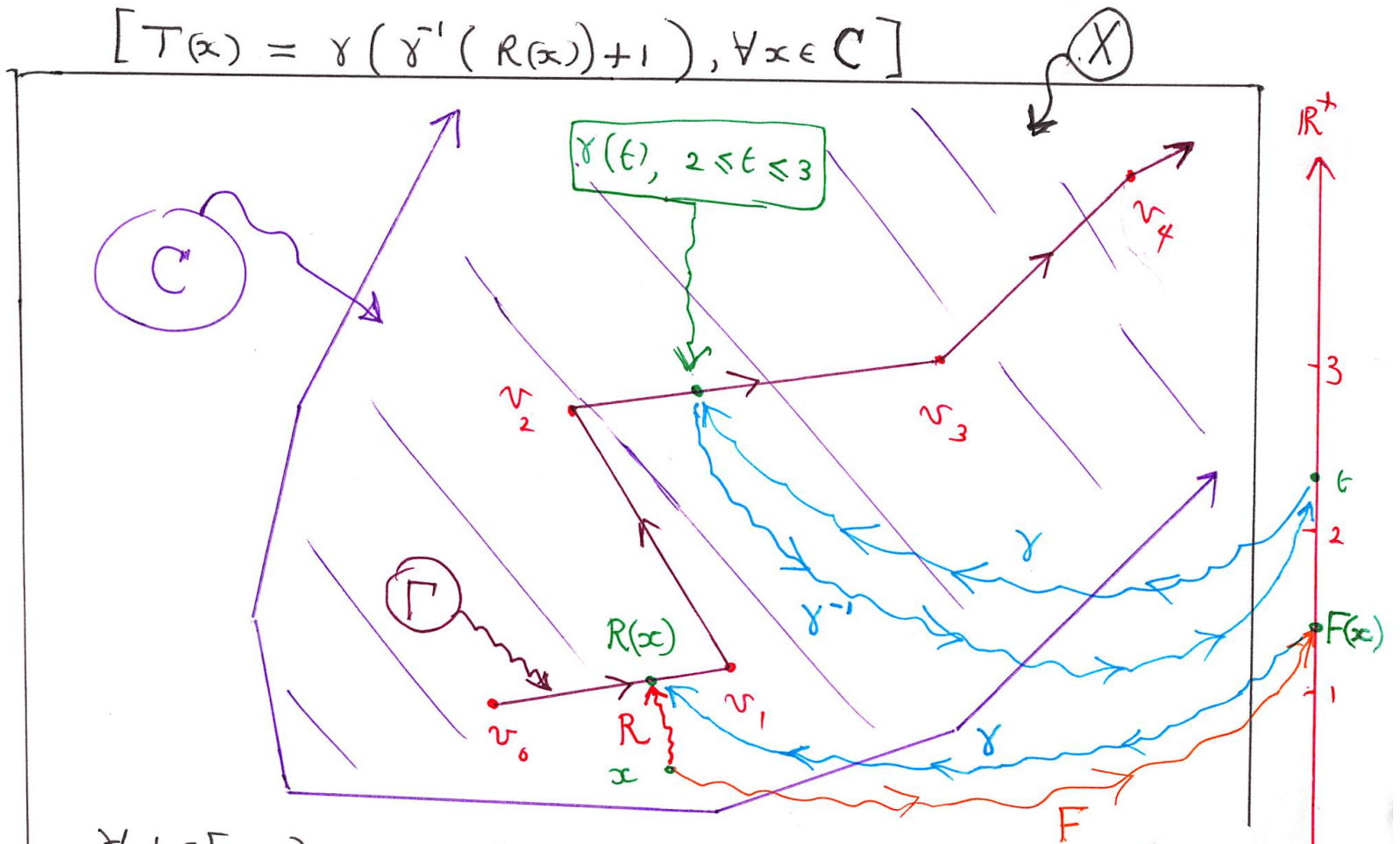
$$\|v_n - v_m\| > a, \text{ for all } n \neq m .$$

$\forall t \geq 0$, let $\langle t \rangle := t - \lfloor t \rfloor$ ($\lfloor t \rfloor :=$ the largest integer $\leq t$), and

$$\gamma(t) := (1 - \langle t \rangle) v_{\lfloor t \rfloor} + \langle t \rangle v_{\lfloor t \rfloor + 1} .$$

γ maps $[0, \infty)$ one-to-one and onto its range Γ , which is a closed subset of C . The map $\gamma^{-1} : \Gamma \longrightarrow [0, \infty)$ is continuous. By Tietze's Extension Theorem, there exists a continuous extension $F : C \longrightarrow [0, \infty)$ of γ^{-1} . The map $T : C \longrightarrow \Gamma \subseteq C$ given by $T : x \mapsto \gamma(F(x) + 1)$ is continuous and fixed point free.

$$[T(x) = \gamma(\gamma^{-1}(R(x)) + 1), \forall x \in C]$$



$$\forall t \in \underbrace{[0, \infty)}_{\mathbb{R}^+}, r_t := \langle t \rangle, n_t := [t], \gamma(t) := (1 - r_t)v_{n_t} + r_t v_{n_t + 1}$$

$[R(x) := \gamma(F(x)), \forall x \in C]$ is a continuous retraction of \$C\$ onto \$\Gamma\$.

$[T: x \mapsto \gamma(F(x) + 1)]$ is a continuous extension of $[\gamma(t) \mapsto \gamma(t + 1)]$.

In summary, we have the following characterization of norm compact, convex subsets of a Banach space in terms of a fixed point property.

Theorem 1.1 ((★) Brouwer, Schauder, Klee). *Let $(X, \|\cdot\|)$ be a Banach space. Let C be a non-empty, closed and convex subset of X . Then T.F.A.E.*

(1) *C is norm compact.*

(2) *Every continuous mapping $f : C \longrightarrow C$ has a fixed point.*

2. COMPLETE METRIC SPACES

Let \mathbb{N} denote the set of all positive integers: $\mathbb{N} = \{1, 2, 3, \dots\}$.

Recall that a metric space (X, d) is *complete* if for every sequence $(x_n)_{n \in \mathbb{N}}$ with $[d(x_n, x_m) \longrightarrow 0, \text{ as } n, m \longrightarrow \infty]$, there exists $z \in X$ with $[d(x_n, z) \longrightarrow 0, \text{ as } n \longrightarrow \infty]$.

Define the *metric space* $X := \{1/2^n : n \in \mathbb{N}\}$, with the usual metric

$$d\left(\frac{1}{2^n}, \frac{1}{2^m}\right) := \left| \frac{1}{2^n} - \frac{1}{2^m} \right|, \text{ for all } m, n \in \mathbb{N}.$$

Note that (X, d) is an *incomplete metric space*. Now consider the mapping $f : X \longrightarrow X : 1/2^n \mapsto 1/2^{n+1}$. Then f is a *strict contraction*:

$$d(f(1/2^n), f(1/2^m)) \leq (1/2) d(1/2^n, 1/2^m), \text{ for all } m, n \in \mathbb{N}.$$

Moreover, f is *fixed point free* on X .

On the other hand, we have the following theorem.

Theorem 2.1 (Banach's Contraction Mapping Theorem (1922)).

Let (X, d) be a complete metric space and the mapping $f : X \longrightarrow X$ be a strict contraction; i.e., for some $k \in [0, 1)$,

$$d(f(x), f(y)) \leq k d(x, y) \text{ , for all } x, y \in X \text{ .}$$

Then f has a fixed point.

The above example and theorem lead to a *characterization of complete metric spaces in terms of a fixed point property.*

Theorem 2.2 (♥). *Let (X, d) be a metric space. T.F.A.E.*

(1) (X, d) is complete.

(2) For all closed non-empty subsets K of X , every strict contraction $f : K \longrightarrow K$ has a fixed point.

Notes. [1] Banach's Contraction Mapping Theorem directly gives us that (1) implies (2) in (♥) above.

[2] (2) implies (1) in (♥) : This theorem is due to Hu (1967). Also see Cobzaş (2016), page 8.

[3] A variation on Hu's proof of (♥) (2) implies (1).

Assume [not (1)]. Let's see why the statement [not (2)] follows.

Since (X, d) is an incomplete metric space, there exists a sequence $(x_n)_{n \in \mathbb{N}}$ in X that is Cauchy, but not convergent. Let the metric space (Z, d) be a completion of (X, d) . Then, there exists $z \in Z \setminus X$ such that $0 < d(x_n, z) \longrightarrow 0$, as $n \longrightarrow \infty$.

There exists a subsequence $(w_j = x_{n_j})_{j \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$ such that

$$d(w_{j+1}, z) < \frac{1}{4} d(w_j, z) , \text{ for all } j \in \mathbb{N} .$$

Let $K := \{w_j : j \in \mathbb{N}\}$. Clearly, K is infinite, since $w_j \neq w_k$ for all $j \neq k$. Also, K is a closed subset of X . Further, it follows from the triangle inequality that

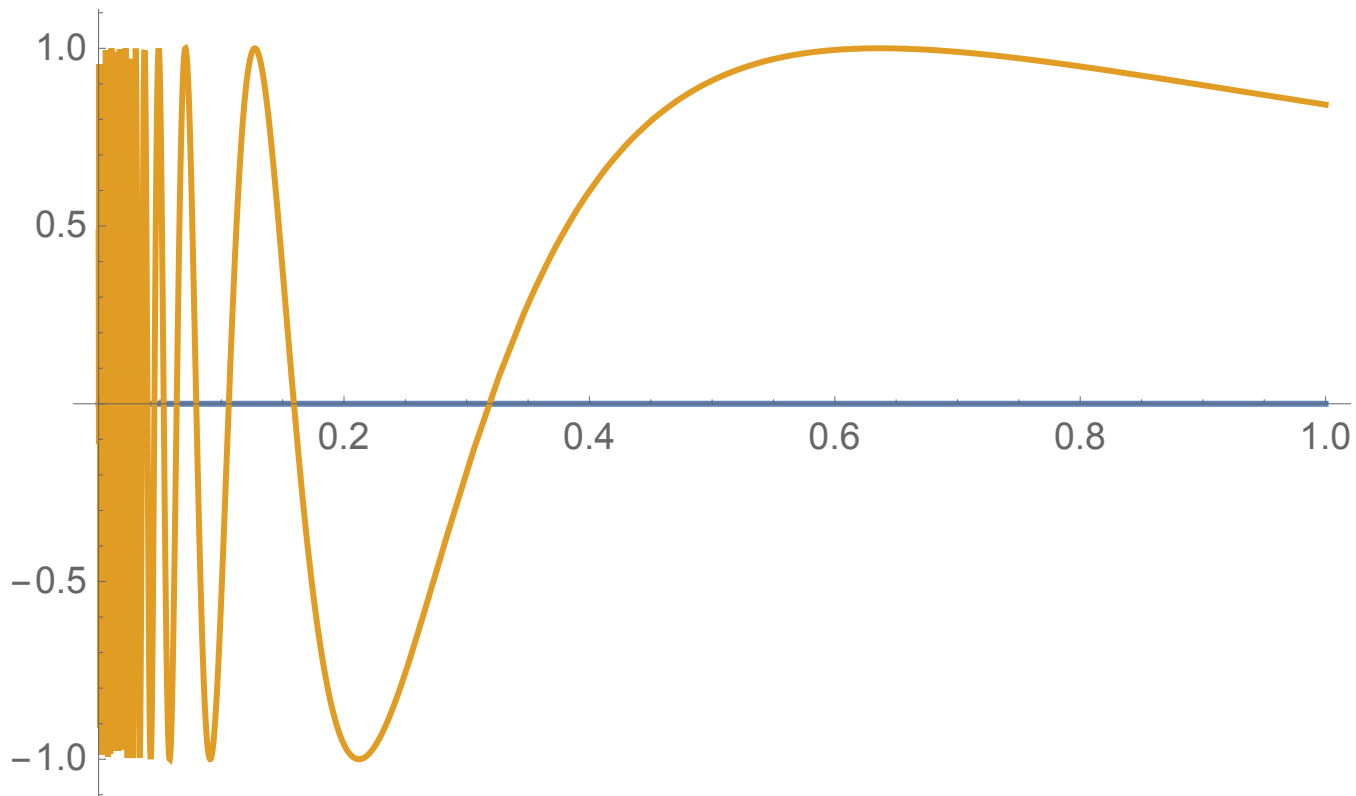
$$d(w_{j+1}, w_{k+1}) \leq \frac{1}{2} d(w_j, w_k) , \text{ for all } j, k \in \mathbb{N} .$$

Thus, the right shift mapping $f : K \longrightarrow K : w_j \mapsto w_{j+1}$ is a *strict contraction that is fixed point free on K* .

Theorem (♥) cannot be improved to resemble Theorem (★): Indeed, by Borwein (1983) there exists an *incomplete* metric space (X, d) on which every strict contraction has a fixed point.

Let d be the usual metric on \mathbb{R}^2 , and let $X \subseteq \mathbb{R}^2$ be given by

$$X := \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) : 0 < x \leq 1 \right\} .$$



On the other hand, Borwein (1983) proved an analogue of Theorem (★) in Banach spaces. (Also see Cobzaş, page 10.)

Theorem 2.3 (♡♡). *Let C be a non-empty, convex subset of a normed linear space $(X, \|\cdot\|)$, with the metric d induced by the norm. Then T.F.A.E.*

(1) (C, d) is complete.

(2) Every strict contraction $f : C \longrightarrow C$ has a fixed point.

Also, Florinskii (1999) proved the following remarkable theorem.

Theorem 2.4. *Let (X, d) be a metric space. Then there exists an equivalent metric σ on X such that every σ -strict contraction $f : X \longrightarrow X$ has a fixed point.*

Note that not every metric space is completely metrizable: e.g., the rational numbers $(\mathbb{Q}, d_{|\cdot|})$ form an incomplete metric space on which every equivalent metric σ is also incomplete.

For more results related to Section 2, see (for example) the references below: [Borwein], [Cobzaş], [Elekes], [Geschke], [Overflow2010], [Xiang], [Goebel-Kirk], [Kirk-Sims], [Piasecki], [Gallagher-L]. Note: [Cobzaş] contains many more references.

For a different kind of converse to Banach's theorem, see (for example): [Bessaga], [Janos], [Meyers].

3. COMPACT METRIC SPACES.

A metric space (X, d) is compact, by definition, if every covering of X by a family of open sets has a finite sub-cover. In metric spaces all the usual variations on compactness (e.g., *sequential compactness*) are equivalent. Moreover, (X, d) is compact if and only if it is complete and totally bounded. Further, a metric space (X, d) is not totally bounded if and only if there exists $r > 0$ and a sequence $(x_n)_{n \in \mathbb{N}}$ in X for which

$$d(x_n, x_m) > r, \text{ for all } n \neq m \text{ in } \mathbb{N} .$$

A mapping $f : (X, d) \longrightarrow (X, d)$ is called *contractive* if

$$d(f(x), f(y)) < d(x, y), \text{ for all } x \neq y \in X .$$

We have the following fixed point theorem of Nemytskiĭ(1936). (Also see Edelstein (1961, 1962).)

Theorem 3.1 (Nemytskiĭ(1936)). *Let (X, d) be a compact metric space. Then every contractive mapping $f : X \longrightarrow X$ has a fixed point.*

Note. (1) On non-compact metric spaces (X, d) , the result may fail. E.g., $X := [1, \infty)$, with its usual metric $d = d_{|\cdot|}$. Then the mapping $f : x \mapsto x + 1/x$ is contractive and fixed point free.

(2) In the “graph($\sin(1/x)$)” example above, the mapping

$$f : (x, \sin(1/x)) \mapsto \left(\frac{x}{1 + 2\pi x}, \sin\left(\frac{1}{x}\right) \right)$$

is contractive and fixed point free.

(3) Also, there exist non-compact metric spaces (X, d) for which the conclusion of Theorem 3.1 is true. Let $X = \{x_n : n \in \mathbb{N}\}$ be any countably infinite set of distinct points x_n . Let d be the *discrete metric on X* ; i.e., $d(x_n, x_m) := 1$, for all $n \neq m$. Then (X, d) is not totally bounded, and so not compact. Moreover, every contractive mapping $f : X \longrightarrow X$ is necessarily a constant mapping; and so has a fixed point.

Further, for every subset K of X (which is necessarily closed), every contractive mapping $f : K \longrightarrow K$ is also constant, and so has a fixed point.

So, a characterization of compact metric spaces in terms of contractive mappings, directly analogous to Theorem (♥) above, is impossible.


However, we may still *characterize compact metric spaces in terms of a contractive fixed point property using the following variation on a theme...*

Theorem 3.2 (). *Let (X, d) be a metric space. T.F.A.E.*

(1) *(X, d) is compact.*

(2) *For all closed non-empty subsets K of X , for all metrics σ on K that are equivalent to d on K , every σ -contractive mapping $f : K \longrightarrow K$ has a fixed point.*

Note. (1) **By Theorem 3.1, we need only to prove (2) implies (1); or equivalently, not(1) implies not(2).**

Proof. () not(1) implies not(2).) Let (X, d) be a non-compact metric space. So, either X is not complete, or X is complete but not totally bounded.

Case 1: X is not complete. By Theorem (\heartsuit), there exists a non-empty closed subset K of X and a strict contraction $f : K \longrightarrow K$ with respect to d , such that f is fixed point free on K . Let $\sigma := d$. Then f is σ -contractive; and we are done.

Case 2: X is complete, but not totally bounded. By replacing d by $d/(1 + d)$, we may W.L.O.G. also assume, that (X, d) is a bounded metric space. Then there exists $r > 0$ and a sequence $(x_n)_{n \in \mathbb{N}}$ in X for which

$$1 > d(x_n, x_m) > r, \text{ for all } n \neq m \text{ in } \mathbb{N}.$$

Let $K := \{x_n : n \in \mathbb{N}\}$. Then K is a closed and infinite subset of X . Let's define the function $\sigma : K \times K \longrightarrow [0, \infty)$ by

$$\sigma(x_n, x_m) := 1 + \frac{1}{2^n} + \frac{1}{2^m}, \text{ for all } n \neq m \in \mathbb{N};$$

and $\sigma(x_n, x_n) := 0$. It is easy to check that σ is a metric on K . Further, σ is equivalent to d on K .

Indeed, for all $n \neq m \in \mathbb{N}$,

$$d(x_n, x_m) < 1 < \sigma(x_n, x_m) < 2 < \frac{2}{r} d(x_n, x_m) .$$

Lastly, consider the shift mapping $f : K \longrightarrow K : x_n \mapsto x_{n+1}$.

Then f is contractive on K . Indeed, for all $x_n \neq x_m$ in K ,

$$\sigma(f(x_n), f(x_m)) = 1 + \frac{1}{2^{n+1}} + \frac{1}{2^{m+1}} < 1 + \frac{1}{2^n} + \frac{1}{2^m} = \sigma(x_n, x_m) .$$

Clearly, f is fixed point free on K . □

4. KIRK, JAGGI AND KASSAY.

A mapping T on a metric space (X, d) is called *nonexpansive* if $d(T(x), T(y)) \leq d(x, y)$, for all $x, y \in X$.

Given a Banach space $(X, \|\cdot\|)$ over \mathbb{R} , its dual space X^* is defined by $X^* := \{ \text{continuous linear functionals } \varphi : X \longrightarrow \mathbb{R} \}$. Define $\|\varphi\|_* := \sup\{|\varphi(x)| : \|x\| \leq 1\}$. Then $(X^*, \|\cdot\|_*)$ is also a Banach space. Now X isometrically embeds into $X^{**} := (X^*)^*$ via the linear map j given by $[(jx)(\varphi) := \varphi(x), \text{ for all } \varphi \in X^*]$, for all $x \in X$. If j maps X onto X^{**} , we call X *reflexive*. **In short:** $X \cong_j X^{**}$.

A Banach space $(X, \|\cdot\|)$ has *normal structure* if for all closed, bounded, convex subsets C of X with $\text{diameter}(C) > 0$, $\text{radius}(C) < \text{diameter}(C)$. Here,

$$\text{diam}(C) := \sup_{x, y \in C} \|x - y\| \quad \text{and} \quad \text{rad}(C) := \inf_{x \in C} \sup_{y \in C} \|x - y\| .$$

In 1965 Kirk proved the following. Also see Browder (1965, 1965) and Göhde (1965).

Theorem 4.1. *Let $(X, \|\cdot\|)$ be a reflexive Banach space with normal structure. Then for every non-empty closed, bounded, convex subset C of X , every $\|\cdot\|$ -nonexpansive mapping $T : C \longrightarrow C$ has a fixed point.*

Examples of reflexive Banach spaces with normal structure are the Lebesgue spaces L^p , $1 < p < \infty$; and more generally: every uniformly convex Banach space.

Jaggi (1983) extended Kirk's theorem in this way:

Theorem 4.2. *Let $(X, \|\cdot\|)$ be a reflexive Banach space with normal structure. Then for every non-empty closed, bounded, convex subset C of X , every $\|\cdot\|$ -Jaggi-nonexpansive mapping $T : C \longrightarrow C$ has a fixed point.*

Note: a mapping $T : C \longrightarrow C$ is called *Jaggi-nonexpansive* if for all non-empty closed, bounded, convex subsets E of C , with $T(E) \subseteq E$, for all $x \in E$,

$$\sup_{y \in E} \|T(x) - T(y)\| \leq \sup_{y \in E} \|x - y\| .$$

In 1986, Kassay proved a converse theorem, that gives us a characterization of reflexive Banach spaces with normal structure *in terms of a fixed point property*.

Theorem 4.3. *Let $(X, \|\cdot\|)$ be a reflexive Banach space. Then T.F.A.E.*

- (1) *X has normal structure.*
- (2) *For every non-empty closed, bounded, convex subset C of X , every Jaggi-nonexpansive mapping $T : C \longrightarrow C$ has a fixed point.*

5. REFLEXIVITY IFF THE FIXED POINT PROPERTY FOR NONEXPANSIVE MAPS... ?

We say a Banach space $(X, \|\cdot\|)$ has the Fixed Point Property for nonexpansive maps on closed, bounded, convex sets [FPP(n.e., c.b.c.)] if for every non-empty closed, bounded, convex subset C of X , every $\|\cdot\|$ -nonexpansive mapping $T : C \longrightarrow C$ has a fixed point.

Kirk's theorem says that if a Banach space $(X, \|\cdot\|)$ is reflexive and has normal structure, then it has the FPP(n.e., c.b.c.). Many nonreflexive Banach spaces fail the FPP(n.e., c.b.c.).

Example (1): $(\ell^1, \|\cdot\|_1)$.

$\ell^1 := \{y = (y_n)_{n \in \mathbb{N}} : y_n \in \mathbb{R}, \text{ \& } \|y\|_1 := \sum_{n=1}^{\infty} |y_n| < \infty\}$.

Consider the scalar sequence $e_n = (0, \dots, 0, 1, 0, \dots, 0, \dots)$,

where the 1 is in position n . Let $C :=$ the closed convex hull of $(e_n)_{n \in \mathbb{N}}$ inside ℓ^1 . Then

$$C := \{x = \sum_{n=1}^{\infty} t_n e_n : \text{each } t_n \geq 0 \text{ and } \sum_{n=1}^{\infty} t_n = 1\} .$$

Define the map $T : C \longrightarrow C : \sum_{n=1}^{\infty} t_n e_n \mapsto \sum_{n=1}^{\infty} t_n e_{n+1}$.

Note that C is closed, bounded and convex, and T is an affine extension of the right shift $e_n \mapsto e_{n+1}$. Also, T is fixed point free and $\|\cdot\|_1$ -nonexpansive.

Example (2): $(c_0, \|\cdot\|_{\infty})$. $\|x\|_{\infty} := \sup_{n \in \mathbb{N}} |x_n|$.

$c_0 := \{x = (x_n)_{n \in \mathbb{N}} : \text{each } x_n \in \mathbb{R}, \& \lim_{n \rightarrow \infty} x_n = 0\}$. **Consider the scalar sequence** $\sigma_n = (1, \dots, 1, 1, 0, \dots, 0, \dots)$, where the last 1 is in position n . Let $K :=$ the closed convex hull of $(\sigma_n)_{n \in \mathbb{N}}$ inside c_0 .

$$K := \{x = \sum_{n=1}^{\infty} t_n \sigma_n : \text{each } t_n \geq 0 \text{ and } \sum_{n=1}^{\infty} t_n = 1\} .$$

Define the map $U : K \longrightarrow K : \sum_{n=1}^{\infty} t_n \sigma_n \mapsto \sum_{n=1}^{\infty} t_n \sigma_{n+1}$.

The set K is closed, bounded and convex, and U is an

affine extension of the right shift $\sigma_n \mapsto \sigma_{n+1}$. Further, U is fixed point free and $\|\cdot\|_\infty$ -nonexpansive.

In 1980 Maurey proved that inside $(L^1[0, 1], \|\cdot\|_1)$, every reflexive subspace X has the FPP(n.e., c.b.c.).

In Dowling-L (1997), we proved that every nonreflexive subspace Y of $(L^1[0, 1], \|\cdot\|_1)$ fails the FPP(n.e., c.b.c.), because it contains a perturbed copy of Example (1) above. (Indeed, each such a Y contains an asymptotically isometric copy of ℓ^1 .) Therefore, we have a characterization of reflexive subspaces of L^1 in terms of a fixed point property.

Theorem 5.1. *Let X be a Banach subspace of $(L^1[0, 1], \|\cdot\|_1)$. Then T.F.A.E.*

(1) X is reflexive.

(2) $(X, \|\cdot\|_1)$ has the FPP(n.e., c.b.c.).

It is still an open question for general Banach spaces:
“If X is reflexive, does $(X, \|\cdot\|)$ have the FPP(n.e., c.b.c.)?”

On the other hand the converse question:

“If a Banach space $(X, \|\cdot\|)$ has the FPP(n.e., c.b.c.), does it follow that X is reflexive?”

was answered *in the negative* by Lin (2008). The P.K. Lin norm, is defined by: for all $x = (x_n)_{n \in \mathbb{N}} \in \ell^1$,

$$\| \|x\| \|_L := \sup_{k \geq 1} \gamma_k \sum_{n=k}^{\infty} |x_n| ;$$

where $(\gamma_k)_k$ is a nondecreasing sequence in $(0, 1)$ with $\lim_k \gamma_k = 1$. Lin (2008) proved that $(\ell^1, \| \cdot \|_L)$ has the FPP(n.e., c.b.c.) for $\gamma_k := \frac{8^k}{1+8^k}$. This condition was extended to every sequence $(\gamma_k)_k$ with $\lim_k \gamma_k = 1$. (See

Fetter (2006) and Hernández-Japón (2010).) Subsequently, many authors have discovered sufficient conditions implying the FPP(n.e., c.b.c.) for equivalent norms on ℓ^1 . (See the references of CDFJLST (2017).)

Fix a nonincreasing sequence $p = (p_n)_n$ in $(1, +\infty)$ with $\lim_n p_n = 1$. We define $X = \mathbb{R} \oplus_{p_1} (\mathbb{R} \oplus_{p_2} (\mathbb{R} \oplus_{p_3} \dots))$, and

$$\nu_p(x) := \left(|x_1|^{p_1} + \left(|x_2|^{p_2} + (|x_3|^{p_3} + \dots)^{p_3} \right)^{p_2/p_3} \right)^{p_1/p_2} \Big)^{1/p_1} .$$

Then X is a Banach space with a *boundedly complete* Schauder basis $(e_n)_n$. By DJLT (1997), $\nu_p(\cdot)$ is equivalent to the usual norm $\|\cdot\|_1$ on ℓ^1 if and only if there exists some $\delta > 0$ such that $p_n/(p_n - 1) \geq \delta \log n$ for all $n \in \mathbb{N}$. **By CDFJLST (2017), each of the Banach spaces $(X, \nu_p(\cdot))$ has the FPP(n.e., c.b.c.).**

Moreover, in CDFJLST, we extend the above results by defining the concept of a *near-infinity concentrated norm* on a Banach space X with a boundedly complete Schauder basis. When $\|\cdot\|$ is such a norm, we prove that $(X, \|\cdot\|)$ has the fixed point property for nonexpansive maps on closed, bounded, convex sets [FPP(n.e., c.b.c.)].

Open Question: An equivalent norm $|||\cdot|||$ on $(\ell^1, \|\cdot\|_1)$ is such that $(\ell^1, |||\cdot|||)$ has the FPP(n.e., c.b.c.) \iff (???)

6. RENORMINGS AND CASCADING NONEXPANSIVE MAPS.

In 2009, Domínguez Benavides proved a theorem analogous to that of Florinskii (1999), discussed above:

Theorem 6.1. *Let $(X, \|\cdot\|)$ be a reflexive Banach space. Then there exists an equivalent norm $\|\cdot\|^\sim$ on X such that $(X, \|\cdot\|^\sim)$ has the FPP(n.e., c.b.c.).*

In 2014, L-Nezir used this result and the Strong James' Distortion Theorems in ℓ^1 and c_0 to prove:

Theorem 6.2. *Let $(X, \|\cdot\|)$ be a Banach space with an unconditional basis, or a Banach lattice. T.F.A.E.*

(1) X is reflexive. (2) *There exists an equivalent norm $\|\cdot\|^\sim$ on X such that for all non-empty closed, bounded, convex subsets C of X , every $\|\cdot\|^\sim$ -cascading nonexpansive mapping $T : C \longrightarrow C$ has a fixed point.*

Here, “*cascading nonexpansive*” means the following. Let $C_0 := C$ and $C_n :=$ the closed convex hull of $T(C_{n-1})$, for all $n \in \mathbb{N}$. Note that $C_0 \supseteq C_1 \supseteq C_2 \supseteq \dots$. This is our “cascade” of sets. We call $T : C \rightarrow C$ **cascading non-expansive** if there exists a sequence $(\lambda_n)_{n \geq 0}$ in $[1, \infty)$ with $\lambda_n \xrightarrow{n} 1$ such that for every integer $n \geq 0$,

$$\|T(x) - T(y)\|^\sim \leq \lambda_n \|x - y\|^\sim, \text{ for all } x, y \in C_n .$$

Note: “nonexpansive” \implies “cascading nonexpansive”, but not conversely. Also, neither “cascading n.e.” nor “asymptotically n.e.” implies the other...

Related recent papers include Domínguez Benavides and Japón (2016), L-Nezir (2017), and L-Nezir-Piasecki (2017).

7. WEAKLY COMPACT CONVEX SETS.

Consider these two Banach spaces, with the supremum norm $\|\cdot\|_\infty$:

$c_0 := \{x = (x_n)_{n \in \mathbb{N}} : \text{each } x_n \in \mathbb{R}, \& \lim_{n \rightarrow \infty} x_n = 0\}$; and
 $c := \{z = (z_n)_{n \in \mathbb{N}} : z_n \in \mathbb{R}, \text{ and } \lim_{n \rightarrow \infty} z_n \text{ exists in } \mathbb{R}\}$.

In 1980, Maurey proved that in c and c_0 , every non-empty weakly compact, convex subset C is such that every $\|\cdot\|_\infty$ -nonexpansive map $T : C \longrightarrow C$ has a fixed point.

In 2003 and 2004, Dowling-L-Turett proved converses in c_0 and c . Combining these results, we can characterize *weakly compact, convex subsets of c_0 and c via fixed point properties.*

Theorem 7.1 (†). *Let C be a non-empty closed, bounded, convex subset of c_0 . T.F.A.E.*

(1) *C is weakly compact.*

(2) *Every $\|\cdot\|_\infty$ -nonexpansive mapping $T : C \longrightarrow C$ has a fixed point.*

Theorem 7.2 (††). *Let K be a non-empty closed, bounded, convex subset of c . T.F.A.E. (1) K is weakly compact.*

(2) *For every non-empty closed, convex subset E of K , every $\|\cdot\|_\infty$ -nonexpansive mapping $U : E \longrightarrow E$ has a fixed point.*

In 2015, Gallagher-L-Popescu proved that Theorem (††) cannot be improved to resemble Theorem (†): [There exists a non-weakly compact c.b.c. subset Γ of c , such that every $\|\cdot\|_\infty$ -nonexpansive map $S : \Gamma \longrightarrow \Gamma$ has a fixed point.] Moreover, Γ is hyperconvex...

Since $(c, \|\cdot\|_\infty)$ is isomorphic to $(c_0, \|\cdot\|_\infty)$, this result also gives us an **equivalent norm** $\|\cdot\|^\sim$ on c_0 such that the analogue of [(2) implies (1)] in Theorem (†) fails.

Note: In (††), DLT1 prove [not(1) implies not(2)]. The fixed point free map U they construct is also contractive and an affine right shift...

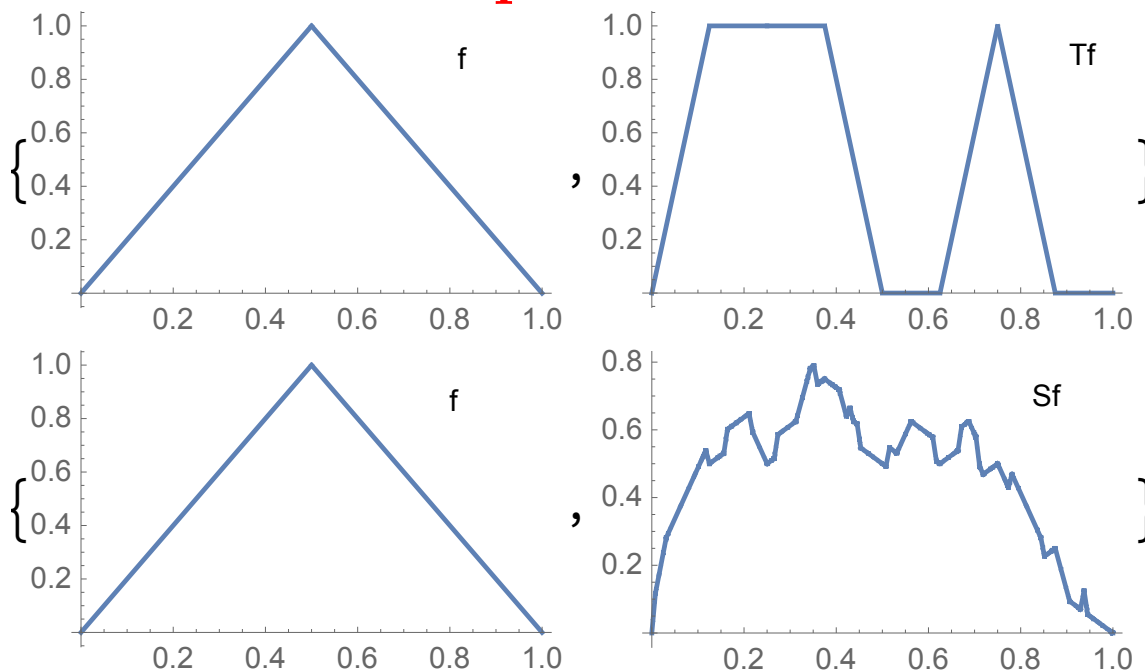
Domínguez Benavides (2012) extended Theorem (†) to *unbounded closed, convex* subsets C of c_0 . Domínguez Benavides and Japón (2016) proved an analogue of Theorem (†) in $(\ell^1, \|\cdot\|_1)$, involving *cascading nonexpansive maps*. Also, Japón-L-Popescu (2017) discovered an analogous result in $(L^1[0, 1], \|\cdot\|_1)$.

THANK YOU !

THANK YOU !

An extra note: there exists a weakly compact, convex subset C of a Banach space $(X, \|\cdot\|)$ on which there is a fixed point free $\|\cdot\|$ -nonexpansive mapping. In 1981 Alspach constructed such an example in $(L^1[0, 1], \|\cdot\|_1)$.

In 2014, Sivek-L-Burns showed that we may replace Alspach's map T by a map S that is $\|\cdot\|_1$ -contractive - and still fixed point free...



Explicitly:

$$C := \left\{ f \in L^1[0, 1] : 0 \leq f \leq 1 \text{ and } \int_0^1 f(x) dx = 1/2 \right\} ,$$

and

$$S := \frac{I}{2} + \frac{T}{4} + \frac{T^2}{8} + \dots .$$

THANK YOU AGAIN !

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