Translation invariant operators in L^p

We say that a bounded linear operator $T : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ is *translation invariant* if $T(\tau_y f) = \tau_y(Tf)$ for all $f \in L^p(\mathbb{R}^n)$ and all $y \in \mathbb{R}^n$, where $(\tau_y f)(x) = f(x+y)$. The following result of Hörmander plays a fundamental role in harmonic analysis since it applies to all convolution type operators.

Theorem (Hörmander 1960). *If $T : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$, $1 \leq p < \infty$, $1 \leq q \leq \infty$ is non-zero and translation invariant, then $q \geq p$.*

The proof is simple and well known. The argument does not generalize to the case of $p = \infty$. However, the argument still works if we replace L^∞ by L^∞_0 which is the subspace of L^∞ consisting of functions that converge to 0 at infinity. In that case Hörmander proved the following result:

Theorem (Hörmander 1960). *If $T : L^\infty_0(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ is non-zero and translation invariant, then $q = \infty$.*

Hörmander calls this result *somewhat incomplete for $p = \infty$*. However, the case of $p = \infty$ has been completely solved by Liu and van Rooij in a paper that is completely unknown (has only one citation in MathSciNet). Their beautiful and surprising result states as follows:

Theorem (Liu and van Rooij 1974). *If $T : L^\infty(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ is non-zero and translation invariant, then $q \geq 2$. On the other hand, there is a non-zero translation invariant operator $T_1 : L^\infty(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. It follows that $T_1 : L^2(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ is bounded for all $2 \leq q \leq \infty$.*

In this talk I will sketch a new proof of this result (joint work with Bownik, Nazarov and Wojtaszczyk).