A SIMPLE PROOF OF EULER'S FORMULA FOR $\zeta(2k)$ AND SOME RATIONAL SERIES REPRESENTATIONS INVOLVING EVEN ARGUMENTS OF THE RIEMANN ZETA FUNCTION.

The Riemann zeta function is defined by

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}, \ \mathrm{Re}\, s>1.$$

Clearly, the series is absolute convergent. It is well-known that ζ is analytic and it has an analytic continuation at $s=1$.

Moreover, the Riemann zeta function has the following integral representation,

$$\zeta(s)=\frac{1}{\Gamma(s)}\cdot\int_0^{\infty}\frac{u^{s-1}}{e^u-1}\,du, \ \mathrm{Re}\, s>1,$$

where $\Gamma(s)=\int_0^{\infty}e^{-t}t^{s-1}dt$, $\mathrm{Re}\, s>0$ is the Euler Gamma function.

In 1734, Leonhard Euler proved (the Basel problem) that

$$\zeta(2)=\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}.$$

Later on (more exactly in 1740), Euler generalized the above result and computed the values of the zeta function at even positive integers. In fact, he proved the following

THEOREM 1.

$$\text{(E)}: \quad \boxed{\zeta(2k)=\sum_{n=1}^{\infty}\frac{1}{n^{2k}}=\frac{(-1)^{k+1}\cdot 2^{2k-1}\cdot B_{2k}}{(2k)!}\cdot\pi^{2k}.}$$

The above theorem says nothing else than the fact that the Riemann zeta function at even arguments is a rational multiple of π^{2k}.

Definition.

The Bernoulli numbers B_n are defined by the power series expansion

$$\frac{t}{e^t-1}=\sum_{n=0}^{\infty}\frac{B_n}{n!}\cdot t^n, \text{ for } |t|<2\pi.$$

Remark.

Note that the function $f(t)=\frac{t}{e^t-1}+\frac{t}{2}$ is even, i.e. $f(t)=f(-t)$, so it follows that $B_1=-\frac{1}{2}$ and $B_k=0$ for all integers $k\geq 3$. The first Bernoulli numbers are easily computed:

K	2	4	6	8	10	12
B_K	$\frac{1}{6}$	$-\frac{1}{30}$	$\frac{1}{42}$	$-\frac{1}{30}$	$\frac{5}{66}$	$-\frac{691}{2730}$

The classical proof (due to Euler) goes along the following lines:

the main ingredient, is an identity (Euler's) for the cotangent function;

$$(C1):\quad \boxed{\pi\cot(\pi x)=\frac{1}{x}+\sum_{k\geq 1}\zeta(2k)x^{2k-1},\ |x|<1.}$$

This formula follows from the following identity:

$$\pi\cot(\pi x)=\frac{1}{x}+\sum_{n\geq 1}\frac{2x}{x^2-n^2}.$$

This is done by expanding the quotient inside the sum sign as a geometric series and interchanging the order of summation.

On the other hand, we have

$$\frac{1}{e^t-1} = \frac{e^{-\frac{t}{2}}}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}} \quad \text{and} \quad -\frac{1}{e^{-t}-1} = \frac{e^{\frac{t}{2}}}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}$$

from which the identity

$$\frac{e^{\frac{t}{2}}+e^{-\frac{t}{2}}}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}} = \frac{2}{t} + 2\sum_{k\geq 1} \frac{B_{2k}\, t^{2k-1}}{(2k)!}$$

follows, using the definition of Bernoulli numbers and the fact that $B_k=0$ for $k\geq 3$ odd. Hence

$$(C2): \quad \boxed{\pi \cot(\pi x) = \pi i \cdot \frac{e^{\frac{2\pi i x}{2}}+e^{-\frac{2\pi i x}{2}}}{e^{\frac{2\pi i x}{2}}-e^{-\frac{2\pi i x}{2}}} = \frac{1}{x} + \sum_{k\geq 1} \frac{(2\pi i)^{2k} B_{2k}}{(2k)!} x^{2k-1}.}$$

By identifying the coefficients in (C1) and (C2), we finally obtain formula (E). ■

In what follows, we present another proof given by de Amo, Carrillo and Hernandez-Sanchez in 2011 which uses the Taylor series for the tangent function and some integration techniques.

Before we start the proof, let us recall the Taylor series for the trigonometric functions tan, cot.

LEMMA. We have the following representations:

$$! \quad \boxed{\begin{aligned} \tan x &= \sum_{n=1}^{\infty} \frac{(-4)^n(1-4^n)\cdot B_{2n}}{(2n)!}\cdot x^{2n-1}, \quad |x|<\frac{\pi}{2}, \text{ and} \\ \cot x &= \sum_{n=0}^{\infty} \frac{(-1)^n\cdot 2^{2n} B_{2n}}{(2n)!} x^{2n-1}, \quad |x|<\frac{\pi}{2}. \end{aligned}}$$

Proof of the lemma.

Let us start with

$$\sum_{n=0}^{\infty} B_n \cdot \frac{t^n}{n!} = \frac{t}{e^t - 1}, \quad |t| < 2\pi.$$

On the left-hand side, we take only the even powers of t, and we get

$$\sum_{n=0}^{\infty} B_n \cdot \frac{t^n}{n!} + \sum_{n=0}^{\infty} B_n \cdot \frac{(-t)^n}{n!} = \frac{t}{e^t - 1} - \frac{t}{e^{-t} - 1}.$$

This gives us

$$2 \sum_{n=0}^{\infty} B_{2n} \cdot \frac{t^{2n}}{(2n)!} = \frac{t(e^{-t} - e^{t})}{(e^t - 1)(e^{-t} - 1)} =$$

$$= \frac{t\left(e^{-\frac{1}{2}t} + e^{\frac{1}{2}t}\right)\left(e^{-\frac{1}{2}t} - e^{-\frac{1}{2}t}\right)}{e^{\frac{1}{2}t}\left(e^{\frac{1}{2}t} - e^{-\frac{1}{2}t}\right)\left(e^{-\frac{1}{2}t} - e^{-\frac{1}{2}t}\right) \cdot e^{-\frac{1}{2}t}}$$

$$= \frac{t\left(e^{\frac{1}{2}t} + e^{-\frac{1}{2}t}\right)}{e^{\frac{1}{2}t} - e^{-\frac{1}{2}t}}.$$

Now, for $\boxed{t = 2ix}$, we have

$$2 \sum_{n=0}^{\infty} B_{2n} \cdot \frac{(2ix)^{2n}}{(2n)!} = \frac{2ix(e^{ix} + e^{-ix})}{e^{ix} - e^{-ix}}$$

or

$$\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 4^n \cdot B_{2n}}{(2n)!} \cdot x^{2n-1} = \frac{i(e^{ix} + e^{-ix})}{e^{ix} - e^{-ix}}.$$

Recall that $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$ and $\cos x = \frac{e^{ix} + e^{-ix}}{2}$

Since $\cot x = \dfrac{\cos x}{\sin x}$, it follows that $\cot x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 4^n \cdot B_{2n}}{(2n)!} \cdot x^{2n-1}$.

On the other hand, we have

$$\cot x - 2\cot(2x) = \cot(x) - \frac{\cot^2(x)-1}{\cot(x)} = \frac{1}{\cot(x)} = \tan x,$$

and this implies that

$$\tan x = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 4^n \cdot B_{2n}}{(2n)!} \cdot x^{2n-1} - 2\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 4^n \cdot 2^{2n-1} \cdot B_{2n}}{(2n)!} \cdot x^{2n-1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 4^n (1-4^n) \cdot B_{2n}}{(2n)!} \cdot x^{2n-1},$$

and we are done. ■

Some remarks.

1) In a similar way, one can deduce a formula for the cosecant function. Indeed, using the identity

$$\cot(x) - \cot(2x) = \cot x - \frac{\cot^2 x - 1}{2\cot x} = \frac{\csc^2 x}{2\cot x} =$$

$$= \frac{1}{2\sin x \cos x} = \csc(2x),$$

and thus

$$\boxed{\begin{aligned} \csc(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 4^n \cdot 2^{2n-1} \cdot B_{2n}}{(2n)!} - \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 4^n \cdot B_{2n}}{(2n)!} x^{2n-1} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (2-4^n)\, B_{2n}}{(2n)!} x^{2n-1} \end{aligned}}$$

(2.) Euler's product formula reads as

(P1): $$\boxed{\sin x = x \cdot \prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right)}$$

and the „cosine" analogue is given by

(P2): $$\boxed{\cos x = \prod_{n=1}^{\infty}\left(1-\frac{4x^2}{(2n-1)^2\pi^2}\right).}$$

In what follows, using formula (P1), we shall derive formula (C1). Indeed, we have:

$$\log(\sin(x)) = \log x + \sum_{n=1}^{\infty}\log\left(1-\frac{x^2}{(n\pi)^2}\right)$$

and taking the derivative, we find that

$$\cot x = \frac{1}{x} + \sum_{n=1}^{\infty}\frac{2x}{x^2-(n\pi)^2}$$

which can be rewritten as

$$\cot x = \frac{1}{x} + \sum_{n=1}^{\infty}\left(\frac{1}{x+n\pi} - \frac{1}{x-n\pi}\right)$$

or equivalently

$$\cot x = \frac{1}{x} + \sum_{n=1}^{\infty}\frac{1}{n\pi}\left(\frac{1}{1+\frac{x}{n\pi}} - \frac{1}{1-\frac{x}{n\pi}}\right) =$$

$$= \frac{1}{x} + \sum_{n=1}^{\infty}\frac{1}{n\pi}\left[\sum_{k=0}^{\infty}(-1)^k\cdot\left(\frac{x}{n\pi}\right)^k - \sum_{k=0}^{\infty}\left(\frac{x}{n\pi}\right)^k\right].$$

Note that in the sum over k, all the even k terms cancel exactly. Thus, all we need to do is sum over odd values of k. Setting $k=2j-1$ and interchanging the order of summation, we derive

$$\cot x = \frac{1}{x} - 2\sum_{n=1}^{\infty} \frac{1}{n\pi} \sum_{j=1}^{\infty} \left(\frac{x}{n\pi}\right)^{2j-1} =$$

$$= \frac{1}{x} - 2\sum_{j=1}^{\infty} \frac{x^{2j-1}}{\pi^{2j}} \sum_{n=1}^{\infty} \frac{1}{n^{2j}}$$

$$= \frac{1}{x} - 2\sum_{j=1}^{\infty} \frac{x^{2j-1}}{\pi^{2j}} \cdot \zeta(2j)$$

and formula (C1) is finally proved. $\blacklozenge$

Now, we are ready to prove Euler's formula:

<u>Proof of the Theorem.</u>

Using the Taylor series expansion for the tangent (see lemma) function, we have that for $|x| \leq \frac{\pi}{2}$ and $|y| < 1$,

$$\int_0^{\frac{\pi}{2}} \tan(xy)\,dx = \int_0^{\frac{\pi}{2}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot 2^{2n}(2^{2n}-1) B_{2n}}{(2n)!} \cdot (xy)^{2n-1}\,dx$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot 2^{2n}(2^{2n}-1) B_{2n}}{(2n)!}\, y^{2n-1} \cdot \int_0^{\frac{\pi}{2}} x^{2n-1}\,dx =$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot \cancel{2^{2n}}(2^{2n}-1) B_{2n}}{(2n)!} \cdot \frac{\pi^{2n}}{(2n)\cancel{2^{2n}}}\, y^{2n-1}.$$

On the other hand, using integration by parts, together with the infinite product formula for the cosine function and Fubini's theorem, we have integration by parts or u-substitution

$$\int_0^{\frac{\pi}{2}} \tan(xy)\,dx = -\frac{\log(\cos(xy))}{y}\Bigg|_0^{\frac{\pi}{2}} = \frac{-\log\left(\cos\left(\frac{\pi y}{2}\right)\right)}{y} =$$

$$= \frac{-\log\left(\prod_{n=1}^{\infty}\left(1-\frac{\left(\frac{\pi y}{2}\right)^2}{\pi^2\left(n-\frac{1}{2}\right)^2}\right)\right)}{y} = \frac{-\log\left(\prod_{n=1}^{\infty}\left(1-\frac{\left(\frac{\pi y}{2}\right)^2}{\pi^2\left(n-\frac{1}{2}\right)^2}\right)\right)}{y}$$

infinite product formula for cosine (P2-formula)

$$= \frac{-\log\left(\prod_{n=1}^{\infty}\left(1-\frac{y^2}{(2n-1)^2}\right)\right)}{y} = \frac{-\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\cdot\frac{(-1)^k}{(2n-1)^{2k}}\cdot y^{2k}}{y}$$

Taylor series expansion for „$\log(1+x)$"

$$= -\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{2k+1}}{k}\cdot\frac{y^{2k-1}}{(2n-1)^{2k}} = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k}\cdot\frac{y^{2k-1}}{(2n-1)^{2k}} =$$

$$= \sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{(2n-1)^{2k}}\cdot\frac{y^{2k-1}}{k}\right).$$

Fubini's theorem

Now, identifying the coefficients in the two series expansions, we obtain:

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^{2k}} = \frac{(-1)^k\cdot(1-4^k)B_{2k}}{(2k)!}\cdot\frac{\pi^{2k}}{2}.$$

Now, we only need to observe that

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} - \sum_{n=1}^{\infty} \frac{1}{(2n)^{2k}} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^{2k}} \quad \text{or equivalently}$$

$$\zeta(2k) - \frac{1}{2^{2k}} \cdot \zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^{2k}} \quad \text{and finally}$$

this gives us

$$\left(1 - \frac{1}{4^k}\right) \zeta(2k) = \frac{(-1)^k \cdot (1-4^k) B_{2k}}{(2k)!} \cdot \frac{\pi^{2k}}{2}$$

and Euler's formula follows immediately. $\square$

Now, in the second part, we would like to compute some rational zeta series involving $\zeta(2k)$.

We would like to compute the following series:

(1) $$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{4^k} = \frac{1}{2}$$

(2) $$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{16^k} = \frac{4-\pi}{8}$$

(3) $$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{k \cdot 4^k} = \log(\pi) - \log 2$$

(4) $$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{k \cdot 16^k} = \log \pi - \frac{3}{2} \log 2$$

It looks like formula (C1) might be really useful here! Let's recall (C1) in the following form:

$$(C1): \quad \boxed{\sum_{k=1}^{\infty} \zeta(2k) \cdot \frac{x^{2k}}{\cancel{k}} = \frac{1}{2}\left(1 - \pi x \cot(\pi x)\right), \quad |x| < 1.}$$

Dividing by x and integrating once, we get

$$(D1): \quad \boxed{\sum_{k=1}^{\infty} \zeta(2k) \cdot \frac{x^{2k}}{k} = \log\left(\frac{\pi x}{\sin(\pi x)}\right), \quad |x| < 1.}$$

The sums (1) and (3) follow immediately for $x = \frac{1}{2}$ in (C1) and (D1) and (2) and (4) follow for $x = \frac{1}{4}$ pluged in (C1) and (D1).

Remark.

Formula (C1) has a deeper connection with Euler's digamma function,

$$\Psi(z) = \frac{d}{dz}\left(\log \Gamma(z)\right) = \frac{\Gamma'(z)}{\Gamma(z)},$$

where $\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1}\, dt$ is the Euler's Gamma function. It's taylor series is given by

$$\boxed{\Psi(1+z) = -\gamma + \sum_{k=2}^{\infty} (-1)^k \cdot \zeta(k) \cdot z^{k-1},}$$

where $\gamma = 1 - \int_1^{\infty} \frac{\{t\}}{t^2}\, dt$ is the Euler-Mascheroni constant.

Also, note that the digamma function satisfies the following equalities:

- <u>duplication formula</u>:

$$\Psi(1-z) - \Psi(z) = \pi \cot(\pi z).$$

- <u>recurrence relation</u>:

$$\Psi(1+z) - \Psi(z) = \frac{1}{z}.$$

Now, we finally have:

$$\sum_{k=1}^{\infty} \zeta(2k) \cdot z^{2k} = \frac{1}{2} z \left[\Psi(1+z) - \Psi(1-z) \right] =$$

$$= \frac{1}{2} z \left[\Psi(z) + \frac{1}{z} - \Psi(1-z) \right]$$

$$= \frac{1}{2} - \frac{\pi}{2} z \cot(\pi z).$$