- $\mathcal{P} = C^\infty(\mathbb{R}^n, \mathbb{C}^m)$ 2π-periodic

- $$\varphi_\xi = \frac{1}{(2\pi)^n} \int_Q \varphi(x) e^{-ix\cdot\xi}\, dx \qquad (*)$$

- <u>Proposition</u>: $\sum_\xi \varphi_\xi e^{ix\cdot\xi}$ converges uniformly to φ.

<u>Proof</u>: Integrate $(*)$ by parts $2k$ times in each variable:

$$(**) \qquad |\varphi_\xi| \le \frac{C_k'}{\xi_1^{2k} \cdots \xi_n^{2k}} \le \frac{C_k}{(1+|\xi|^2)^k} \qquad \text{for all } \xi \in \mathbb{Z}^n.$$

- <u>Lemma 1</u>: $\sum_\xi \frac{1}{(1+|\xi|^2)^k} < \infty$ when $k \ge [\frac{n}{2}] + 1$.

  <u>Proof</u>: Integral test. ∎

Take $k \ge [n/2] + 1$ in $(**)$ and $\sum_\xi \varphi_\xi e^{ix\cdot\xi}$ converges absolutely uniformly to a continuous function Φ.
Put $\Psi = \varphi - \Phi$ and $\Psi_\xi = 0$ $\forall \xi$. $\int_Q \Psi \cdot t = 0$ for trig. polynomials. Stone-Weierstrass $\Longrightarrow$ $\Psi \equiv 0$. $\Phi = \varphi$. ∎

Derivatives & Fourier Series

- Integration by Parts gives

$$D^\alpha \varphi(x) = \sum_\xi \xi^\alpha \varphi_\xi e^{ix\cdot\xi} \qquad \left(\text{where } D^\alpha \varphi = \left(\frac{1}{i}\right)^{[\alpha]} \tilde{D}^\alpha \varphi \right)$$

i.e.

$$(D^\alpha \varphi)_\xi = \xi^\alpha \varphi_\xi$$

- Parseval : $\displaystyle\int_Q |\varphi|^2 = (2\pi)^n \sum_\xi |\varphi_\xi|^2$

- Therefore

$$\|D^\alpha \varphi\|^2 = \sum_\xi \xi^{2\alpha} |\varphi_\xi|^2$$

- Finally, we get the motivation for Sobolev Spaces

$$\boxed{c \sum_\xi (1+|\xi|^2)^t |\varphi_\xi|^2 \le \sum_{[\alpha]=0}^{t} \|D^\alpha \varphi\|^2 \le \sum_\xi (1+|\xi|^2)^t |\varphi_\xi|^2}$$

- $\mathcal{S}$ – sequences in $\mathbb{C}^m$ indexed by $\mathbb{Z}^n$.

- $H_s = \left\{ u \in \mathcal{S} : \sum_{\xi} (1+|\xi|^2)^s |u_\xi|^2 < \infty \right\}$

- From Schwarz Inequality

$$\left| \sum_{\xi} (1+|\xi|^2)^{\frac{s+t}{2}} u_\xi \cdot v_\xi \right|^2 \le \left(\sum_{\xi} (1+|\xi|^2)^s |u_\xi|^2 \right) \left(\sum_{\xi} (1+|\xi|^2)^t |u_\xi|^2 \right)$$

- $\langle u, v \rangle_s = \sum_{\xi} (1+|\xi|^2)^s u_\xi \cdot v_\xi$, $\quad \|u\|_s = \langle u, u \rangle_s^{1/2}$

makes H_s a Hilbert Space

- $\mathcal{P} \subset H_s$. Indeed, $|\varphi_\xi| \le \frac{C_k}{(1+|\xi|^2)^k}$ where $C_k = C_k(\varphi, D^\alpha \varphi : [\alpha] \le k)$

and $\sum_{\xi} \frac{1}{(1+|\xi|^2)^k} < \infty$ for $k \ge [n/2] + 1$.

- Fact: the more differentiable a function is, the faster the Fourier coefficients $|\varphi_\xi|$ decay, the higher order Sobolev space φ belongs.

<u>Sobolev Lemma:</u> If $u \in H_t$, $t \geq [\frac{n}{2}] + 1 + m$, then $D^\alpha u$ converges uniformly for $[\alpha] \leq m$. Thus each $u \in H_t$ for $t \geq [\frac{n}{2}]+1+m$ corresponds to a function $\sum_\xi u_\xi e^{ix\cdot\xi}$ of class C^m.

<u>Proof:</u> By induction. $m=0$: we have $t \geq [\frac{n}{2}]+1$ so,

$$\sum_{\xi < N} |u_\xi| = \sum_{|\xi|<N} (1+|\xi|^2)^{-t/2} (1+|\xi|^2)^{t/2} |u_\xi|$$

$$\leq \Big(\sum_{|\xi|<N} (1+|\xi|^2)^{-t} \Big)^{\frac{1}{2}} \|u_t\| \qquad \text{(Hölder-Schwarz)}$$

$$\leq C_t \|u_t\| \qquad \text{(Lemma 1)}$$

Now suppose $m \neq 0$. $D^\alpha u \in H_{t-[\alpha]}$ for $[\alpha] \leq m$ so the series for $D^\alpha u$ converges uniformly by the case $m=0$. It's a well-known result in analysis that this implies $u \in C^m$.

Lemma (Difference Quotients) Let $u \in H_s$ and $\|u^h\|_s \leq k$ for some k, all nonzero $h \in \mathbb{R}^n$. Then $u \in H_{s+1}$. (Converse also holds)

Proof: Let u_N be the truncation of u at N. Need to prove these are uniformly bounded. By hypothesis,

$$\|u_N^h\|_s^2 = \sum_{|\xi| \leq N} (1+|\xi|^2)^s |u_\xi|^2 \left| \frac{e^{ih\cdot\xi} - 1}{|h|} \right|^2 \leq k^2.$$

$\downarrow$ ~~Assume~~ (Let $h = te_i$ and $t \to 0$)

$$\sum_{|\xi| \leq N} (1+|\xi|^2)^s |u_\xi|^2 |\xi_i|^2 \leq k^2$$

So

$$\|u_N\|_{s+1} = \sum_{|\xi| \leq N} (1+|\xi|^2)^{s+1} |u_\xi|^2$$

$$= \sum_{|\xi| \leq N} (1+|\xi|^2)^s |u_\xi|^2 + \sum_{i=1}^{n} |\xi_i|^2 (1+|\xi|^2)^s |u_\xi|^2$$

$$\leq \|u\|_s^2 + nk^2$$

So $u_N \in H_{s+1}$. ■

It's even easier to show the converse.

Partial Differential Operators

We consider operators of the form

$$L = P_l(D) + \dots + P_0(D)$$

where $P_j(D)$ is an $m \times m$ matrix each entry of which is a homogeneous differential operator of order j, $\sum_{|\alpha|=j} a_\alpha D^\alpha$ where the a_α are $C^\infty(\mathbb{R}^n, \mathbb{C})$. Say that L is elliptic if

$$P_l(w) = \Big(\sum_{|\alpha|=l} a^{ij}_\alpha w_1^{\alpha_1} \cdots w_n^{\alpha_n} \Big)_{ij} \quad \text{is nonsingular}$$

matrix for each x.

Fundamental Inequality: Let L be elliptic on $\mathcal{O}$ of order l and $s \in \mathbb{Z}$. There is a constant $c > 0$ s.t.

$$\|u\|_{s+l} \le c \left(\|Lu\|_s + \|u\|_s \right)$$

for $u \in H_{s+l}$.

Proof of Fundamental Inequality ① $L = L_0$ is constant, with only leading term $P_l(D)$ nonzero.

$$\|L_0\varphi\|_s^2 = \sum_\xi |P_l(\xi)\varphi_\xi|^2(1+|\xi|^2)^s$$
$$\geq \sum_\xi c|\xi|^{2l}|\varphi_\xi|^2(1+|\xi|^2)^s$$

Hence

$$(\|L_0\varphi\|_s + \|\varphi\|_s)^2 \geq \|L_0\varphi\|_s^2 + \|\varphi\|_s^2$$
$$\geq \sum_\xi |\varphi_\xi|^2(1+|\xi|^2)^s(1 + \text{const}\,|\xi|^{2l})$$
$$\geq \text{const}\sum_\xi |\varphi_\xi|^2(1+|\xi|^2)^{s+l}$$
$$= \text{const}\,\|\varphi\|_{s+l}^2$$

② Consider general elliptic operator of order l on $\mathcal{P}$. $p \in \mathbb{R}^n$. We'll find $U \ni p$ such that F.E. holds for $\varphi \in \mathcal{P}$ with support in U. Let L_0 be the constant coefficient elliptic operator of order l determined by L at p. Then

$$\|\varphi\|_{s+l} \leq \text{const}\,(\|L_0\varphi\|_s + \|\varphi\|_s)$$
$$\leq \text{const}\,(\|L\varphi\|_s + \|(L_0 - L)\varphi\|_s + \|\varphi\|_s)$$
$$= \text{const}\,(\|L\varphi\|_s + \|\tilde{L}\varphi\|_s + \|\varphi\|_s)$$

Let $\tilde{L}$ be a periodic extension of $L - L_0$ off a small neighborhood U of p. Let $\text{supp}(\varphi) \subset U$

(♡ Long boring estimate)

$$\leq \text{Const}\,\{\|L\varphi\|_s + \tfrac{1}{2}\|\varphi\|_{s+l} + \text{const}\,\|\varphi\|_{s+l-1} + \text{const}\,\|\varphi\|_s$$
$$\leq \text{const}\,\|L\varphi\|_s + \tfrac{3}{4}\|\varphi\|_{s+l} + \text{const}\,\|\varphi\|_s$$

We used Peter-Paul: $\|u\|_t \le \varepsilon \|u\|_{t''} + c(\varepsilon)\|u\|_{t''}$ $(t' < t < t'')$.
for $u \in H_{t''}$.

③ Let $p_1, \dots, p_k$ be such that $U_{p_1}, \dots, U_{p_k}$ cover the torus. Let $\omega_1, \dots, \omega_k$ be a square partition of unity. $\sum_i \omega_i^2 = 1$. Let $\varphi \in \mathcal{P}$. Now

$$\|\varphi\|_{s+l}^2 = \langle \varphi, \varphi \rangle_{s+l} = \left\langle \sum_i \omega_i^2 \varphi, \varphi \right\rangle$$

(unproven lemma)

$$\le \sum_i \langle \omega_i \varphi, \omega_i \varphi \rangle_{s+l} + \text{const} \|\varphi\|_{s+l} \|\varphi\|_{s+l-1}$$

$$\le \text{const} \sum_i \|L\omega_i \varphi\|^2 + \text{const}\|\varphi\|_s^2 + \text{const}\|\varphi\|_{s+l}\|\varphi\|_{s+l-1}$$

$$\le \text{const} \sum_i \langle L(\omega_i^2 \varphi), L\varphi \rangle + \text{const}\|\varphi\|_s^2 + \text{const}\|\varphi\|_{s+l}\|\varphi\|_{s+l-1}$$

$$= \text{const}\|L\varphi\|_s^2 + \text{const}\|\varphi\|_s^2 + \text{const}\|\varphi\|_{s+l}\|\varphi\|_{s+l-1}$$

$$\le \text{const}\|L\varphi\|_s^2 + \text{const}\|\varphi\|_s^2 + \tfrac{1}{2}\|\varphi\|_{s+l}^2 + \text{const}\|\varphi\|_{s+l-1}$$

$$\le \text{const}\|L\varphi\|_s^2 + \text{const}\|\varphi\|_s^2 + \tfrac{3}{4}\|\varphi\|_{s+l}^2 + \text{const}\|\varphi\|_s$$

$$\|\varphi\|_{s+l}^2 \le \text{const}\|L\varphi\|_s^2 + \text{const}\|\varphi\|_s^2$$

∎

Reduction to Periodic Case

Let $C^\infty = C^\infty(\mathbb{R}^n, \mathbb{C}^m)$. $C_0^\infty = C_0^\infty(\mathbb{R}^n, \mathbb{C}^m)$. $C_0^\infty(V) = C_0^\infty(V, \mathbb{C}^m)$. For $u, v \in C_0^\infty$ let $\langle u, v\rangle = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} u \cdot v$. If $V \subset \bar{V} \subset Q$ then $C_0^\infty(V) \subset \mathcal{P}$. $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_0$. Let L be an elliptic partial differential operator of order l on C^∞. L has formal adjoint L^* on C_0^∞ w.r.t. $\langle \cdot, \cdot \rangle$ by integration by parts:

$$L_{ij} = \sum a_{ij}^\alpha D^\alpha \Rightarrow L_{ij}^* = \sum D^\alpha \bar{a}_{ji}^\alpha$$

Let $p \in \mathbb{R}^n$. Let $\tilde{L}$ be a periodic extension of L away from p, say, $\tilde{L} = L$ on $V \ni p$. By approximation,

(1) $\langle \tilde{L} u, \varphi \rangle = \langle u, L^* \varphi \rangle$ for $u \in H_s$, $\varphi \in C_0^\infty(V)$

Assume $V \subset \bar{V} \subset Q$. $u, v \in H_s$. Say $u = v$ on V if $\langle u - v, \varphi \rangle_0 = 0$ for $\varphi \in C_0^\infty(V)$. By (1) $Lu = Lv$ if $u = v$ on V provided L has support in V. Let $\langle \cdot, \cdot \rangle'$ denote the inner product on $E^p(M)$. We will prove

Given a C^∞ p-form f on M and bounded linear functional

(2) $l' : E^p(M) \to \mathbb{R}$ such that $l'(\Delta^* \varphi) = \langle f, \varphi \rangle'$ for every $\varphi \in E^p(M)$, there exists a C^∞ p-form u on M such that $l'(t) = \langle u, t \rangle'$ for $t \in E^p(M)$.

<u>Step 1</u>: Localization. Let U be a coordinate neighborhood of M, $\gamma : M \to \mathbb{R}^n$ onto. Now differentiable p-forms become $C^\infty(\mathbb{R}^n, \mathbb{R}^m)$ functions $(m = \binom{n}{p})$. The Laplacian Δ induces an <u>elliptic</u> partial differential operator of order 2 on C^∞. Call the induced map L. Let L^* be the formal adjoint (with respect to $\langle \cdot, \cdot \rangle$). Transfer $\langle \cdot, \cdot \rangle'$ to Euclidean Space via γ^{-1*} to obtain an inner product, still denoted $\langle \cdot, \cdot \rangle'$, on $C_0^\infty(\mathbb{R}^n, \mathbb{C}^m)$. There is a self-adjoint matrix of smooth functions A such that

$$\langle \varphi, \psi \rangle' = \langle \varphi, A\psi \rangle \qquad \forall \varphi, \psi \in C_0^\infty.$$

For $\varphi \in C_0^\infty$ $\quad L^*\varphi = A\Delta^* A^{-1}\varphi$. Indeed,

$$\langle L^*\varphi, \psi \rangle = \langle \varphi, L\psi \rangle = \langle A^{-1}\varphi, L\psi \rangle'$$
$$= \langle \Delta^* A^{-1}\varphi, \psi \rangle' = \langle A\Delta^* A^{-1}\varphi, \psi \rangle.$$

Define l on C_0^∞ by $l(\varphi) = l'(A^{-1}\varphi)$. We shall show

(3) If $p \in \mathbb{R}^n$ there is W_p of p and $u_p \in \mathcal{P}$ s.t. $l(t) = \langle u_p, t \rangle$ for $t \in C_0^\infty(W_p)$.

For motivation let's show (3) $\Rightarrow$ $\boxtimes$. Indeed, the u_p must agree on overlaps so piece together to give $u \in C^\infty$. Then

$$l'(\varphi) = l(A\varphi) = \langle u, A\varphi \rangle = \langle u, \varphi \rangle' \text{ as desired.}$$

Partition of unity $\{W_p\}$

Step 2: Proof of (3). Fix $p\in\mathbb{R}^n$, Q' a 2π-cube containing p, $V\ni p$, $p\in V\subset\bar V\subset Q'$ and let $\tilde l = l|_{C_0^\infty(V)}$. $\tilde l : C_0^\infty(V)\to\mathbb{C}$ is bounded since

$$|\tilde l(\varphi)| = |l(\varphi)| = |l'(A^{-1}\varphi)| \le \text{Const}\,\|A^{-1}\varphi\|' \le \text{const}\,\|\varphi\|. \quad (\varphi\in C_0^\infty(V))$$

~~Now observe $\tilde l(L^*\varphi) = \langle f,\varphi\rangle$ $\forall\varphi\in C_0^\infty(V)$~~. $\tilde l$ extends to a bounded linear functional on H_0. So there is $\tilde u\in H_0$ s.t.

$$\tilde l(t) = \langle \tilde u, t\rangle_0 \qquad \forall t\in H_0.$$

Our task: Show $\tilde u$ agrees with an element of $\mathcal{P}$ on a nbd of p. Set up

$$p\in O\subset\bar O\subset\cdots\subset O_2\subset\bar O_2\subset O_1\subset\bar O_1\subset O_0\subset\bar O_0\subset V\subset\bar V\subset Q$$

~~For~~ $n\ge 1$ choose $\omega_n\in C^\infty_c$. $\bar O_n<\omega_n<O_{n-1}$. Let $v_1=\omega_1\tilde u\in H_0$. Then

$$\tilde L v_1 = \tilde L\omega_1\tilde u = \omega_1 L\tilde u + M_1\tilde u$$

$$= \underbrace{\underset{C^\infty}{\omega_1 f} + \underset{H_{-1}}{M_1\tilde u}}_{\Downarrow\,\boxed{\text{regularity}}}$$

$M_1 = \tilde L\omega_1 - \omega_1\tilde L$ is order 1
$\tilde L$ agrees with L on O_0
is periodic

$$\begin{aligned}
\langle\omega_1\tilde L\tilde u-\omega_1 f,\varphi\rangle_0 &= \langle\omega_1\tilde L\tilde u,\varphi\rangle_0-\langle\omega_1 f,\varphi\rangle_0\\
&= \langle\tilde L\tilde u,\omega_1\varphi\rangle_0-\langle f,\omega_1\varphi\rangle_0\\
&= \langle\tilde u, L^*\omega_1\varphi\rangle-\tilde l(L^*\omega_1\varphi)\\
&= \tilde l(L^*\omega_1\varphi)-\tilde l(L^*\omega_1\varphi)=0
\end{aligned}$$

$v_1\in H_1$. Now let $v_2=\omega_2\tilde u$. Then

$$\tilde L v_2 = \omega_2 L\tilde u + M_2\tilde u$$

$$\underbrace{\underset{C^\infty}{} \quad \underset{H_0}{}}_{\Downarrow\,\boxed{\text{regularity}}}$$

$M_2 = \tilde L\omega_2-\omega_2\tilde L$ has spt in O_1 and $\tilde u = v_1$ on O_1

$v_2\in H_2$

...