PANSU DIFFERENTIABILITY

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1. Carnot Groups

1.1. **Carnot-Carathéodory Metrics.** Let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $X = (X_1, \dots, X_m)$ be a family of vector fields with locally Lipschitz continuous coefficients on Ω. Vector fields will be written

$$X_j(x) = (a_{1j}(x), \dots, a_{nj}(x))$$

for $j = 1, \dots, m$, where a_{ij} are locally Lipschitz on Ω. We will write the coefficients a_{ij} in the $n \times m$ matrix $\mathcal{A} = \mathrm{col}[X_1, \dots, X_m]$.

Definition 1.1. A Lipschitz curve $\gamma : [0, T] \to \Omega$ is *X-admissible* if there exists a measurable function $h : [0, T] \to \mathbb{R}^m$ such that

(i) $\dot{\gamma}(t) = \mathcal{A}(\gamma(t))h(t) = \sum_{j=1}^m h_j(t)X_j(\gamma(t))$ for a.e. $t \in [0, T]$,
(ii) $h \in L^\infty([0, T])$.

The curve γ is *X-subunit* if it is X-admissible and $\|h\|_\infty \leq 1$.

Remark 1.2. X-admissible is horizontal

Define $d : \Omega \times \Omega \to [0, \infty]$ by

$$d(x, y) = \inf\{T \geq 0 \mid \gamma : [0, T] \to \Omega \text{ is } X\text{-subunit such that } \gamma(0) = x \text{ and } \gamma(T) = y\}.$$

If the above set is empty we define $d(x, y) = \infty$.

Proposition 1.3. *If $d(x, y) < \infty$ for all $x, y \in \Omega$ then (Ω, d) is a metric space.*

The metric space (Ω, d) is called the Carnot-Carathéodory metric space.

2. Pansu Differentiability

Let $\mathbb{G} = (\mathbb{R}^n, \cdot, \delta_\lambda, d)$ and $\bar{\mathbb{G}} = (\mathbb{R}^{\bar{n}}, \bar{\cdot}, \bar{\delta}_\lambda, \bar{d})$ be two Carnot groups. A map $\varphi : \mathbb{G} \to \bar{\mathbb{G}}$ is a *homogeneous homomorphism* if φ is a group homomorphism and $\varphi(\delta_\lambda(x)) = \bar{\delta}_\lambda(\varphi(x))$ for all $x \in \mathbb{G}$ and $\lambda > 0$. A map $f : \mathbb{G} \to \bar{\mathbb{G}}$ is Lipschitz if there exists a constant $M > 0$ such that $\bar{d}(f(x), f(y)) \leq M d(x, y)$ for all $x, y \in \mathbb{G}$. In $\mathbb{G} = \mathbb{R}^n$ we fix the Lebesgue measure and denote by $|E|$ the measure of a measurable set $E \subseteq \mathbb{G}$.

Definition 2.1. A map $f : \mathbb{G} \to \bar{\mathbb{G}}$ is *Pansu-differentiable* (or *differentiable*) at $x \in \mathbb{G}$ if for all $y \in \mathbb{G}$ there exists

$$Df_x(y) = \lim_{t \to 0} \bar{\delta}_{1/t}(f(x)^{-1} f(x \delta_t(y))),$$

and the convergence is uniform with respect to y. The map $Df_x : \mathbb{G} \to \bar{\mathbb{G}}$ is the differential of f at x.

The main theorem in this section is the Pansu differentiability of Lipschitz mappings between Carnot groups. This theorem has interesting consequences which we will state and prove after the proof of this theorem.

Theorem 2.2. *Let $f : \mathbb{G} \to \bar{\mathbb{G}}$ be a Lipschitz map. Then Df_x exists for a.e. $x \in \mathbb{G}$ and is a homogeneous homomorphism.*

Proposition 2.3. *If $Df_x(y)$ exists then there also exists $Df_x(\delta_\lambda(y))$ for all $\lambda > 0$ and*

$$Df_x(\delta_\lambda(y)) = \bar{\delta}_\lambda Df_x(y).$$

Proof. Indeed, for a fixed $\lambda > 0$ and $t > 0$,

$$\bar{\delta}_{1/t}(f(x)^{-1} f(x \delta_t(\delta_\lambda(y)))) = \bar{\delta}_\lambda \bar{\delta}_{1/\lambda t}(f(x)^{-1} f(x \delta_{\lambda t}(y))),$$

by the dilation properties. Thus,

$$\begin{aligned} Df_x(\delta_\lambda(y)) &= \lim_{t \to 0} \bar{\delta}_{1/t}(f(x)^{-1} f(x \delta_t(\delta_\lambda(y)))) \\ &= \bar{\delta}_\lambda \lim_{s \to 0} \bar{\delta}_{1/s}(f(x)^{-1} f(x \delta_s(y))) = \bar{\delta}_\lambda Df_x(y). \end{aligned}$$

□

If $f : \mathbb{G} \to \bar{\mathbb{G}}, x, y \in \mathbb{G}$ and $t > 0$ define

$$R(x, y; t) = \bar{\delta}_{1/t}(f(x)^{-1} f(x \delta_t(y))).$$

Proposition 2.4. *Let $f : \mathbb{G} \to \bar{\mathbb{G}}$ be a Lipschitz map. If for some $y_1, y_2 \in \mathbb{G}$ the derivatives $Df_x(y_1)$ and $Df_x(y_2)$ exist for a.e. $x \in \mathbb{G}$, then there also exists $Df_x(y_1y_2)$ for a.e. $x \in \mathbb{G}$ and*

$$Df_x(y_1y_2) = Df_x(y_1)Df_x(y_2).$$

Proof. By Proposition 2.3, we can assume that $d(y_1, 0) = d(y_2, 0) = 1$. Let Ω be an arbitrary open subset of $\mathbb{G}$ with finite Lebesgue measure and fix $\eta > 0$. By our assumptions, the mappings $x \mapsto Df_x(y_1)$ and $x \mapsto Df_x(y_2)$ are defined almost everywhere on $\mathbb{G}$ and therefore they define two measurable functions from $\mathbb{G}$ to $\bar{\mathbb{G}}$. Also, $R(x, y_2; t) \to Df_x(y_2)$ as $t \to 0$ for a.e. $x \in \mathbb{G}$. By Lusin and Egorov Theorems, there exists a compact set $K \subseteq \Omega$ such that

(i) $|\Omega \setminus K| < \eta$,
(ii) For any $x \in K$, $Df_x(y_1)$ and $Df_x(y_2)$ exist and they are continuous at x,
(iii) $R(x, y_2; t) \to Df_x(y_2)$ as $t \to 0$ uniformly on K.

If we prove the claim for all $x \in K$ we are done. Fix $x \in K$. We have to show that $\lim_{t\to 0} R(x, y_1y_2; t)$ exists and $Df_x(y_1y_2) = Df_x(y_1)Df_x(y_2)$. Using the fact that δ_λ and $\bar{\delta}_\lambda$ are group automorphisms and by "adding and subtracting" the term $f(x\delta_t(y_1))$ we find

$$\begin{aligned} R(x, y_1y_2; t) &= \bar{\delta}_{1/t}(f(x)^{-1}f(x\delta_t(y_1y_2))) = \bar{\delta}_{1/t}(f(x)^{-1}f(x\delta_t(y_1)\delta_t(y_2))) \\ &= \bar{\delta}_{1/t}(f(x)^{-1}f(x\delta_t(y_1))f(x\delta_t(y_1))^{-1}f(x\delta_t(y_1)\delta_t(y_2))) \\ &= \bar{\delta}_{1/t}(f(x)^{-1}f(x\delta_t(y_1)))\bar{\delta}_{1/t}(f(x\delta_t(y_1))^{-1}f(x\delta_t(y_1)\delta_t(y_2))) \\ &= R(x, y_1; t)R(x\delta_t(y_1), y_2; t). \end{aligned} \tag{2.1}$$

We know that $R(x, y_1; t) \to Df_x(y_1)$. So we have to show that $R(x\delta_t(y_1), y_2; t) \to Df_x(y_2)$.

Let $\epsilon > 0$ be chosen arbitrarily. Then by (iii) there exists $\delta > 0$ such that

$$\bar{d}(R(z, y_2; t), Df_z(y_2)) < \epsilon$$

for all $z \in K$ as long as $0 < t < \delta$.

If $x\delta_t(y_1) \in K$ for all $0 < t < \delta'$ and some $0 < \delta' < \delta$ then by (ii) there exists $0 < \delta'' < \delta'$ such that $\bar{d}(Df_{x\delta_t(y_1)}(y_2), Df_x(y_2)) < \epsilon$ when $0 < t < \delta''$ and

$$\begin{aligned} \bar{d}(R(x\delta_t(y_1), y_2; t), Df_x(y_2)) &\leq \bar{d}(R(x\delta_t(y_1), y_2; t), Df_{x\delta_t(y_1)}(y_2)) \\ &+ \bar{d}(Df_{x\delta_t(y_1)}(y_2), Df_x(y_2)) < 2\epsilon. \end{aligned} \tag{2.2}$$

This shows that

$$Df_x(y_1y_2) = \lim_{t\to 0} R(x, y_1y_2; t) = Df_x(y_1)Df_x(y_2).$$

However, in general $x\delta_t(y_1) \notin K$. Let $B(x,r)$ a C-C ball centered at x with radius r. By the Lebesgue differentiation theorem in doubling metric measure spaces, for a.e. $x \in K$ we have

$$\lim_{r\to 0} \frac{|B(x,r) \setminus K|}{|B(x,r)|} = 0.$$

Let $\lambda(t) = \mathrm{dist}(x\delta_t(y_1), K) = d(x\delta_t(y_1), \bar{x}(t))$ for some $\bar{x}(t) \in K$ and define $\bar{y_1} = \delta_{1/t}(x^{-1}\bar{x}(t))$ so that $\bar{x}(t) = x\delta_t(\bar{y_1}(t))$. By Proposition (1.7.3)

$$d(x\delta_t(y_1), x) = d(\delta_t(y_1), 0) = td(y_1, 0) = t,$$

and consequently, $B(x\delta_t(y_1), \lambda(t)) \subseteq B(x, t + \lambda(t)) \setminus K$. Let $Q \geq n$ be the homogeneous dimension of $\mathbb{G}$. By Proposition (1.7.7)

$$\frac{|B(x\delta_t(y_1), \lambda(t))|}{|B(x, t+\lambda(t))|} = \frac{(\lambda(t))^Q |B(0,1)|}{(t+\lambda(t))^Q |B(0,1)|} = \left(\frac{\lambda(t)}{t+\lambda(t)}\right)^Q,$$

and consequently

$$\left(\frac{\lambda(t)}{t+\lambda(t)}\right)^Q = \frac{|B(x\delta_t(y_1), \lambda(t))|}{|B(x, t+\lambda(t))|} \leq \frac{|B(x, t+\lambda(t)) \setminus K|}{|B(x, t+\lambda(t))|}.$$

Notice that $\lambda(t) = \mathrm{dist}(x\delta_t(y_1), K) \leq d(x\delta_t(y_1), x) = t$. Hence

$$0 \leq \lim_{t\to 0} \left(\frac{\lambda(t)}{t+\lambda(t)}\right)^Q \leq \lim_{t\to 0} \frac{|B(x, t+\lambda(t)) \setminus K|}{|B(x, t+\lambda(t))|} = 0,$$

which implies

$$\lim_{t\to 0} \frac{\lambda(t)}{t} = 0. \tag{2.3}$$

We have

$$\lambda(t) = d(x\delta_t(y_1), \bar{x}(t)) = d(x\delta_t(y_1), x\delta_t(\bar{y_1}(t))) = d(\delta_t(y_1), \delta_t(\bar{y_1}(t))) = td(y_1, \bar{y_1}(t)),$$

and from (2.3) it follows that

$$\lim_{t\to 0} d(y_1, \bar{y_1}(t)) = 0. \tag{2.4}$$

We already showed in (2.1) that

$$R(x, y_1y_2; t) = R(x, y_1; t)R(x\delta_t(y_1), y_2; t).$$

Our goal is to show that $R(x\delta_t(y_1), y_2; t)$ converges to $Df_x(y_2)$. Notice that the point $x\delta_t(y_1)$ does not belong to K. So it has to be projected onto the set K in order to apply the argument in (2.2). Write

$$\begin{aligned} R(x\delta_t(y_1), y_2; t) &= \bar{\delta}_{1/t}(f(x\delta_t(y_1))^{-1} f(x\delta_t(\bar{y_1}(t)))) \\ &\quad \bar{\delta}_{1/t}(f(x\delta_t(\bar{y_1}(t)))^{-1} f(x\delta_t(\bar{y_1}(t))\delta_t(y_2))) \\ &\quad \bar{\delta}_{1/t}(f(x\delta_t(\bar{y_1}(t))\delta_t(y_2))^{-1} f(x\delta_t(y_1)\delta_t(y_2))) =: R_1(t)R_2(t)R_3(t). \end{aligned}$$

We claim that $\lim_{t\to 0} R_1(t) = \lim_{t\to 0} R_3(t) = 0$. Let $M > 0$ be the Lipschitz constant of f and notice that

$$\begin{aligned} \bar{d}(R_1(t), 0) &= \bar{d}(\bar{\delta}_{1/t}(f(x\delta_t(y_1))^{-1} f(x\delta_t(\bar{y_1}(t)))), 0) \\ &= \bar{d}(\bar{\delta}_{1/t}(f(x\delta_t(\bar{y_1}(t)))), \bar{\delta}_{1/t}(f(x\delta_t(y_1)))) \\ &= \frac{1}{t}\bar{d}(f(x\delta_t(\bar{y_1}(t))), f(x\delta_t(y_1))) \\ &\leq \frac{M}{t} d(x\delta_t(\bar{y_1}(t)), x\delta_t(y_1)) \\ &= \frac{M}{t} d(\delta_t(\bar{y_1}(t)), \delta_t(y_1)) = Md(\bar{y_1}(t), y_1), \end{aligned}$$

and analogously

$$\begin{aligned} \bar{d}(R_3(t), 0) &= \bar{d}(\bar{\delta}_{1/t}(f(x\delta_t(\bar{y_1}(t))\delta_t(y_2))^{-1} f(x\delta_t(y_1)\delta_t(y_2))), 0) \\ &= \bar{d}(\bar{\delta}_{1/t}(f(x\delta_t(y_1)\delta_t(y_2))), \bar{\delta}_{1/t}(f(x\delta_t(\bar{y_1}(t))\delta_t(y_2)))) \\ &= \frac{1}{t}\bar{d}(f(x\delta_t(y_1)\delta_t(y_2)), f(x\delta_t(\bar{y_1}(t))\delta_t(y_2))) \\ &\leq \frac{M}{t} d(x\delta_t(y_1)\delta_t(y_2), x\delta_t(\bar{y_1}(t))\delta_t(y_2)) \\ &= \frac{M}{t} d(x\delta_t(y_1 y_2), x\delta_t(\bar{y_1}(t) y_2)) \\ &= \frac{M}{t} d(\delta_t(y_1 y_2), \delta_t(\bar{y_1}(t) y_2)) = Md(y_1 y_2, \bar{y_1}(t) y_2). \end{aligned}$$

So, (2.4) implies that both $R_1(t)$ and $R_3(t)$ converge to 0 in $\bar{\mathbb{G}}$. Consider now $R_2(t)$ and notice that

$$R_2(t) = R(x\delta_t(\bar{y_1}(t)), y_2; t).$$

Since $x\delta_t(\bar{y_1}(t)) \in K$ for all t and by (2.4), the argument in (2.2) does apply and hence

$$\lim_{t\to 0} R_2(t) = Df_x(y_2).$$

Therefore,

$$\lim_{t\to 0} R(x\delta_t(y_1), y_2; t) = Df_x(y_2),$$

and finally

$$Df_x(y_1y_2) = Df_x(y_1)Df_x(y_2).$$

□

Let $\mathbb{G} = (\mathbb{R}^n, \cdot, \delta_\lambda, d)$ be a Carnot group and assume that $X = (X_1, \ldots, X_m)$ is a system of generators for the Lie algebra of the group such that $X_j(0) = e_j$. We denote by $\mathcal{A} = \mathrm{col}[X_1, \ldots, X_m]$ the $n \times m$ matrix of the coefficients of the vector fields.

In the following lemma, we heavily use the fact that the underlying manifold of a Carnot group can always be chosen to be $\mathbb{R}^n$ for some $n \in \mathbb{N}$ and therefore take advantage the exponential coordinates. In this situation, the Carnot group $\mathbb{G}$ restricted to the first m-dimensional subspace acts like the regular $\mathbb{R}^m$. This is the case for example for the Heisenberg group $\mathbb{H}^1$ and the 2-dimensional xy-plane.

Lemma 2.5. *Let $\gamma : [0,1] \to \mathbb{G}$ be a Lipschitz curve. Then γ is X-admissible and if $h \in L^\infty([0,1])$ is its vector of canonical coordinates then*

$$\lim_{t\to 0} \delta_{1/t}(\gamma(s)^{-1}\gamma(s+t)) = (h_1(s), \ldots, h_m(s), 0, \ldots, 0)$$

for a.e. $s \in [0,1]$.

Proof. By abuse of notation we identify h with $(h_1, \ldots, h_m, 0, \ldots, 0)$ whenever necessary. By Proposition 1.3.3, γ is X-admissible and $\dot\gamma(s) = \mathcal{A}(\gamma(s))h(s)$ for a.e. $s \in [0,1]$. Define

$$E = \{s \in [0,1] : \dot\gamma(s) = \mathcal{A}(\gamma(s))h(s) \text{ exists and } s \text{ is a Lebesgue point of } h\}.$$

Clearly E is of full measure. Let $s \in E$ and assume, without loss of generality, that $s = 0$. Since the statement is translation invariant we may also assume that $\gamma(0) = 0$. We have to prove that

$$\lim_{t\to 0} \delta_{1/t}(\gamma(t)) = (h_1(0), \ldots, h_m(0), 0, \ldots, 0).$$

Recall that if the coordinate x_i has degree d_i then we can write

$$\delta_{1/t}(\gamma(t)) = \left(\frac{\gamma_1(t)}{t^{d_1}}, \frac{\gamma_2(t)}{t^{d_2}}, \ldots, \frac{\gamma_n(t)}{t^{d_n}}\right),$$

and we write $xy = P(x, y) = x + y + Q(x, y)$. By formula (1.7.83) for a.e. $t \in [0, 1]$

$$\dot{\gamma}(t) = \sum_{j=1}^{m} h_j(t) X_j(\gamma(t)) = \sum_{j=1}^{m} h_j(t) \sum_{i=1}^{n} \frac{\partial P_i}{\partial y_j}(\gamma(t), 0) e_i = \sum_{i=1}^{n} \sum_{j=1}^{m} h_j(t) \frac{\partial P_i}{\partial y_j}(\gamma(t), 0) e_i \tag{2.5}$$

We begin with $i = 1, 2, \ldots, m$. Since the degree of the coordinate x_i is equal to 1, we have to show that

$$\lim_{t \to 0} \frac{\gamma_i(t)}{t} = \dot{\gamma}_i(0) = h_i(0).$$

We know that $Q_i = 0$ by Lemma 1.7.2 (iv). Thus, for all $1 \leq i, j \leq m$,

$$\frac{\partial P_i}{\partial y_j}(\gamma(t), 0) = \delta_{ij},$$

and consequently,

$$\dot{\gamma}_i(t) = h_i(t),$$

for a.e. $t \in [0, 1]$. Since $0 \in E$, we have

$$\dot{\gamma}_i(0) = \lim_{t \to 0} \frac{\gamma_i(t)}{t} = \frac{1}{t} \int_0^t h_i(s)\ ds = h_i(0),$$

for all $i = 1, 2, \ldots, m$.

Now, fix $i = m + 1, \ldots, n$ and assume that the i^{th} coordinate has degree $k \geq 2$ and that the claim has been proved for the degrees $1, 2, \ldots, k - 1$. If we denote by $\bar{Q}_i(x, y)$ the sum of the monomials in $Q_i(x, y)$ in which y appears linearly then by (2.5) we can write

$$\dot{\gamma}_i(t) = \sum_{j=1}^{m} h_j(t) \frac{\partial P_i}{\partial y_j}(\gamma(t), 0) = \sum_{j=1}^{m} h_j(t) \frac{\partial Q_i}{\partial y_j}(\gamma(t), 0) = \bar{Q}_i(\gamma(t), h(t)).$$

It follows from Lemma 1.7.2 (v) that $Q_i(\gamma(t), h(t))$ depends only on the coordinates of $\gamma(t)$ and $h(t)$ with degrees strictly less than k. Moreover, since $\bar{Q}_i$ is homogeneous of degree k and it is the sum of the monomials in $Q_i(x, y)$ in which y appears linearly, each monomial in $\bar{Q}_i(\gamma(t), h(t))$ contains the components $\gamma_1(t), \ldots, \gamma_{i-1}(t)$ homogeneously of degree $k-1$. ($h(t)$ is the second component which plays the role of y.) Thus

$$\bar{Q}_i(\delta_{1/s}(\gamma(s)), h(s)) = \frac{1}{s^{k-1}} \bar{Q}_i(\gamma(s), h(s)).$$

Since $\dot{\gamma_i}(t) = \bar{Q}_i(\gamma(t), h(t))$, we have

$$\begin{aligned}
\left|\frac{\gamma_i(t)}{t^k}\right| &\leq \frac{1}{t^k}\int_0^t \left|\bar{Q}_i(\gamma(s), h(s))\right| ds \\
&= \frac{1}{t}\int_0^t \left|\frac{1}{t^{k-1}}\bar{Q}_i(\gamma(s), h(s))\right| ds \\
&\leq \frac{1}{t}\int_0^t \left|\frac{1}{s^{k-1}}\bar{Q}_i(\gamma(s), h(s))\right| ds \\
&= \frac{1}{t}\int_0^t \left|\bar{Q}_i(\delta_{1/s}(\gamma(s)), h(s))\right| ds.
\end{aligned}$$

By the inductive hypothesis, the j^{th} component of $\delta_{1/t}(\gamma(t))$ converges to $h_j(0)$ for any coordinate j with degree less than or equal to $k-1$. Therefore,

$$\limsup_{t\to 0}\left|\frac{\gamma_i(t)}{t^k}\right| \leq \left|\bar{Q}_i(h(0), h(0))\right|.$$

But $\bar{Q}_i(h(0), h(0)) = 0$ by Lemma 1.7.2 (iv) and the proof is complete. □

Remark 2.6. Let $V = \{\lambda e_j : \lambda \in \mathbb{R} \text{ and } j = 1, \ldots, m\}$. Since the Lie algebra of the group is nilpotent and stratified then it follows that there exists $r \in \mathbb{N}$ such that for every $y \in \mathbb{G}$ there exists $y_1, \ldots, y_r \in V$ such that $y = y_1 \ldots y_r$.

Theorem 2.7 (Pansu-Rademacher Theorem)**.** *Let $f : \mathbb{G} \to \bar{\mathbb{G}}$ be a Lipschitz map. Then $Df_x : \mathbb{G} \to \bar{\mathbb{G}}$ exists for a.e. $x \in \mathbb{G}$ and is a homogeneous homomorphism.*

Proof. Fix $1 \leq j \leq m$ and write $\hat{x}_j = (x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_n)$. So, for any $x \in \mathbb{G}$, the curve $\gamma_{\hat{x}_j} : \mathbb{R} \to \bar{\mathbb{G}}$ defined by

$$\gamma_{\hat{x}_j}(t) = f(\exp(tX_j)(\hat{x}_j))$$

is Lipschitz. Indeed, if $\gamma_j : \mathbb{R} \to \mathbb{G}$ is a solution of the equation $\dot{\gamma_j}(s) = X_j(\gamma_j(s))$ with the initial condition $\gamma_j(0) = 0$, then we have $\exp(X_j) = \gamma_j(1)$ and

$$\exp(tX_j)(\hat{x}_j) = \hat{x}_j \exp(tX_j) = \hat{x}_j\gamma(t).$$

Thus, for any $s, t \in \mathbb{R}$

$$\begin{aligned}
\bar{d}(\gamma_{\hat{x}_j}(s), \gamma_{\hat{x}_j}(t)) &= \bar{d}(f(\exp(sX_j)(\hat{x}_j)), f(\exp(tX_j)(\hat{x}_j))) \\
&\leq Md(\exp(sX_j)(\hat{x}_j), \exp(tX_j)(\hat{x}_j)) \\
&= Md(\hat{x}_j\gamma(s), \hat{x}_j\gamma(t)) \\
&= Md(\gamma(s), \gamma(t)) \leq ML|s-t|.
\end{aligned}$$

Here, M is the Lipschitz constant of f. Notice that the curve γ is an L-Lipschitz mapping because it is an integral curve.
So, by Lemma 2.5, the curve $\gamma_{\hat{x}_j}$ is Pansu differentiable at a.e. $t \in \mathbb{R}$. Let $E_j = \{x \in \mathbb{G} : \gamma_{\hat{x}_j}$ is differentiable at $x_j\}$ and define $E = \bigcap_{j=1}^m E_j$. By Fubini theorem, $|\mathbb{G} \setminus E| = 0$. We will show that f is differentiable at almost every point x in E.

Fix $x \in E$ and since the statement is translation invariant assume without loss of generality that $x = 0$. Let $K = \partial B(0,1) = \{v \in \mathbb{G} : d(v,0) = 1\}$. If $v \in K$ we have to prove that there exists

$$Df_0(v) = \lim_{t \to 0} R(0, v; t) = \lim_{t \to 0} \bar{\delta}_{1/t}(f(0)^{-1} f(\delta_t(v)))$$

and that the convergence is uniform for $v \in K$. Since $\bar{\mathbb{G}}$ with its Carnot-Carathéodory metric $\bar{d}$ is a complete metric space it is enough to show that for all $\epsilon > 0$ there exist $\delta > 0$ such that

$$\sup_{v \in K} \bar{d}(R(0, v; s), R(0, v; t)) < (1 + 2M)\epsilon$$

for all $0 < s, t < \delta$.

Since K is compact, we can find $v_1, \ldots, v_k \in K$ such that $K \subseteq \bigcup_{i=1}^k B(v_i, \epsilon)$. Fix a v_i and denote it by v. By Remark 2.6 we can write $v = y_1 y_2 \ldots y_r$ where each y_i is of the form λe_j for some $\lambda \in \mathbb{R}$ and $j = 1, 2, \ldots, m$. Without loss of generality we can also assume $\lambda = 1$. Now, if $\gamma(t) = f(\exp(tX_j)(0))$ then

$$Df_0(y_i) = \lim_{t \to 0} \bar{\delta}_{1/t}(f(0)^{-1} f(\delta_t(y_i))) = \lim_{t \to 0} \bar{\delta}_{1/t}(\gamma(0)^{-1}\gamma(t))$$

exists for all $i = 1, 2, \ldots, r$ because $0 \in E$. Hence, by Proposition 2.4, $Df_0(v)$ exists and

$$Df_0(v) = Df_0(y_1) \ldots Df_0(y_r).$$

Therefore, there exists $\delta > 0$ such that

$$\sup_{i=1,\ldots,k} \bar{d}(R(0, v_i; s), R(0, v_i; t)) < \epsilon$$

for all $0 < s, t < \delta$. If $v \in K$ then there exists v_i such that $d(v, v_i) < \epsilon$ and

$$\begin{aligned} \bar{d}(R(0, v; s), R(0, v; t)) &\leq \bar{d}(R(0, v; s), R(0, v_i; s)) + \bar{d}(R(0, v_i; s), R(0, v_i; t)) \\ &+ \bar{d}(R(0, v_i; t), R(0, v; t)) \\ &\leq (1 + 2M)\epsilon. \end{aligned}$$

Indeed,

$$\begin{aligned}\bar{d}(R(0,v;s),R(0,v_i;s)) &= \bar{d}(\bar{\delta}_{1/s}(f(0)^{-1}f(\delta_s(v))),\bar{\delta}_{1/s}(f(0)^{-1}f(\delta_s(v_i))))\\ &= \frac{1}{s}\bar{d}(f(0)^{-1}f(\delta_s(v)),f(0)^{-1}f(\delta_s(v_i)))\\ &= \frac{1}{s}\bar{d}(f(\delta_s(v)),f(\delta_s(v_i)))\\ &\leq \frac{M}{s}d(\delta_s(v),\delta_s(v_i))\\ &= Md(v,v_i) < M\epsilon.\end{aligned}$$

Similarly,

$$\bar{d}(R(0,v;t),R(0,v_i;t)) < M\epsilon.$$

Therefore, we showed that for all $\epsilon > 0$ there exist $\delta > 0$ such that

$$\sup_{v\in K}\bar{d}(R(0,v;s),R(0,v;t)) < (1+2M)\epsilon$$

for all $0 < s,t < \delta$.

Now we have to prove the homomorphism.

The proof is complete. □

Corollary 2.8. *There is no biLipschitz embedding of any noncommutative Carnot group* $\mathbb{G}$ *into any Euclidean space* $\mathbb{R}^k$.

Proof. Assume, to the contrary that there exists a biLipschitz map $f:\mathbb{G}\to\mathbb{R}^k$. Then by the Pansu-Rademacher Theorem, f is differentiable at almost every $x\in\mathbb{G}$. Let $x\in\mathbb{G}$ be such that $df_x:\mathbb{G}\to\mathbb{R}^k$ exists. We claim that df_x is also biLipschitz. Indeed, since f is biLipschitz, there exist constants $0<m<M<\infty$ such that

$$md(z_1,z_2)\leq|f(z_1)-f(z_2)|\leq Md(z_1,z_2),$$

for all $z_1,z_2\in\mathbb{G}$. Here, d is the Carnot-Carathéodory metric on $\mathbb{G}$.
Fix $y_1,y_2\in\mathbb{G}$ and $t>0$. We have

$$md(x\delta_t(y_2^{-1}y_1),x)\leq|f(x\delta_t(y_2^{-1}y_1))-f(x)|\leq Md(x\delta_t(y_2^{-1}y_1),x),$$

which implies

$$mtd(y_1,y_2)\leq|f(x\delta_t(y_2^{-1}y_1))-f(x)|\leq Mtd(y_1,y_2).$$

Hence,

$$md(y_1,y_2)\leq\frac{|f(x\delta_t(y_2^{-1}y_1))-f(x)|}{t}\leq Md(y_1,y_2).$$

Sending t to 0 implies

$$md(y_1, y_2) \leq |df_x(y_2^{-1}y_1)| \leq Md(y_1, y_2).$$

Notice that $df_x : \mathbb{G} \to \mathbb{R}^k$ is a homogeneous homomorphism. So, we have

$$md(y_1, y_2) \leq |df_x(y_1) - df_x(y_2)| \leq Md(y_1, y_2).$$

This means that df_x is a biLipschitz map and therefore injective. Since $\mathbb{G}$ is a noncommutative group, there exist $w, z \in \mathbb{G}$ such that $[w, z] \neq 0$. However,

$$df_x([w, z]) = df_x(wzw^{-1}z^{-1}) = df_x(w) + df_x(z) - df_x(w) - df_x(z) = 0,$$

which is a contradiction with the injectivity of df_x. Hence, there is no biLipschitz embedding of any noncommutative Carnot group into any Euclidean space. □

Remark 2.9. Notice that the Heisenberg group $\mathbb{H}^n$ is a noncommutative Carnot group. Therefore, the above corollary implies that there is no biLipschitz embedding of any Heisenberg group into any Euclidean space.