Metric Currents

Defintions Let X be a locally compact metric space.

$$\mathcal{D}(X) = \{f \in \mathrm{Lip}\,(X) : f \text{ has compact support}\}$$

$$\mathrm{Lip}_{K,l}(X) = \{f \in \mathrm{Lip}_l(X) : \mathrm{spt}(f) \subset K\}$$

Notice $\mathcal{D}(X) = \bigcup \mathrm{Lip}_{K,l}(X)$. Say…

$$f_j \to f \text{ in } \mathcal{D}(X)$$

if and only if

- f_j belong to some fixed $\mathrm{Lip}_{K,l}(X)$
- $f_j \to f$ pointwise (hence uniformly) on X

Say…

$$\pi_j \to \pi \text{ in } \mathrm{Lip}_{\mathrm{loc}}(X)$$

if and only if

- For compact $K \subset X$ there is a constant l_K with $\mathrm{Lip}\left(\pi_j|_K\right) \leq l_K$
- $\pi_j \to \pi$ pointwise (hence locally uniformly) on X

Definitions Let $\mathcal{D}^n(X) = \mathcal{D}(X) \times \left[\mathrm{Lip}_{\mathrm{loc}}(X)\right]^n$

Let $T: \mathcal{D}^n(X) \to \mathbb{R}$ be a function satisfying the following properties

1. Multilinearity in the $n+1$ arguments

2. Continuity in the product topology

3. Locality: let $(f, \pi_1, \ldots, \pi_n) \in \mathcal{D}^n(X)$ and suppose some π_i is constant on a neighborhood of $\mathrm{spt}(f)$. Then
$$T(f, \pi) = 0$$

If T satisfies these properties, we call it an n-dimensional metric current on X.

Denote by $\mathcal{D}_n(X)$ the space of these objects.

Endow $\mathcal{D}_n(X)$ with the locally convex weak topology.

$$T_k \to T \text{ if } T_k(f, \pi) \to T(f, \pi) \text{ for all } (f, \pi) \in \mathcal{D}^n(X)$$

Example A submanifold $M^{(m)}$ of a Riemannian manifold V induces an m-current $[\![M]\!] \in \mathcal{D}_m(V)$

$$[\![M]\!](f, g_1, \dots, g_m) = \int_M f dg_1 \wedge \dots \wedge dg_m$$

More generally, if we have in addition a function $u \in L^1_{\mathrm{loc}}(V)$, there is an induced current $[u] \in \mathcal{D}_m(V)$

$$[u](f, g_1, \dots, g_m) = \int_M uf\, dg_1 \wedge \dots \wedge dg_m$$

(Non-) Example Let $X = \mathbb{R}$. We ask whether the dirac mass δ_0 induces a 1-current.

In the classical theory, δ_0 is a current

$$[\delta_0](f, g) = f(0)g'(0)$$

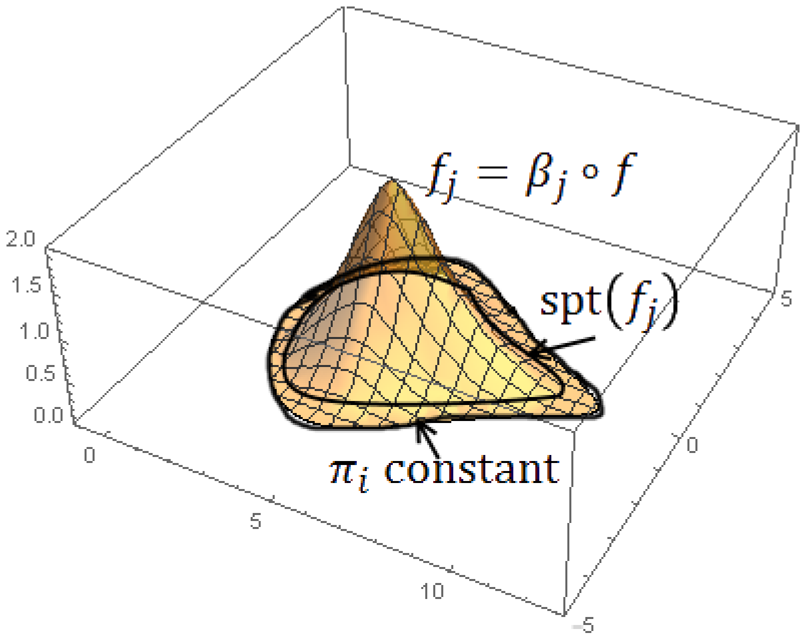
But in the theory of metric currents, δ_0 is not a current because g is merely Lipschitz and may not have a derivative at 0.

This is a serious problem. It seems we have no hope for compactness of $\mathcal{D}_m(X)$.

Lemma (Strict Locality of Metric Currents)

$$T(f, \pi_1, \dots, \pi_m) = 0 \text{ whenever some } \pi_i \text{ is constant on } \mathrm{spt}(f).$$

Proof Replace f with $f_j = \beta_j \circ f$



$$\beta_j(s) = \max(0, s - 1/j)$$

By locality, $T(f_j, \pi_1, \dots, \pi_m) = 0$

↓

$T(f, \pi_1, \dots, \pi_m) = 0$

Lemma (Lang 2.2) Suppose $T\colon [\mathcal{D}(X)]^{m+1} \to \mathbb{R}$ satisfies the conditions of a metric current with $\mathcal{D}(X)$ in place of $\mathrm{Lip}_{\mathrm{loc}}(X)$. Then T extends uniquely to a current in $\mathcal{D}_m(X)$.

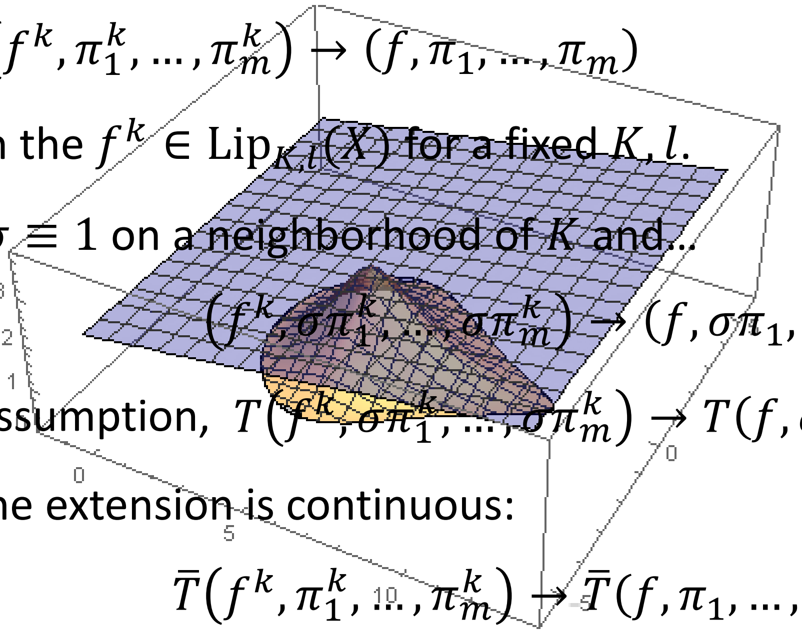
Remark Thus, $T \in \mathcal{D}_m(X)$ is determined by its values on $[\mathcal{D}(X)]^{m+1} \subset \mathcal{D}^m(X)$.

Proof Let T be given as in the hypotheses. Define, for $(f, \pi_1, \dots, \pi_m) \in \mathcal{D}^m(X) = \mathcal{D}(X) \times \left[\mathrm{Lip}_{\mathrm{loc}}(X)\right]^m$

$$\bar{T}(f, \pi_1, \dots, \pi_m) = T(f, \sigma\pi_1, \dots, \sigma\pi_m)$$

Where $\sigma \in \mathcal{D}(X)$ with $\sigma \equiv 1$ on a neighborhood of $\mathrm{spt}(f)$. This is independent of the choice of σ by locality.

The three axioms are now easy to check. For example, continuity:



Let $\left(f^k, \pi_1^k, \dots, \pi_m^k\right) \to (f, \pi_1, \dots, \pi_m)$

Then the $f^k \in \mathrm{Lip}_{K,l}(X)$ for a fixed K, l.

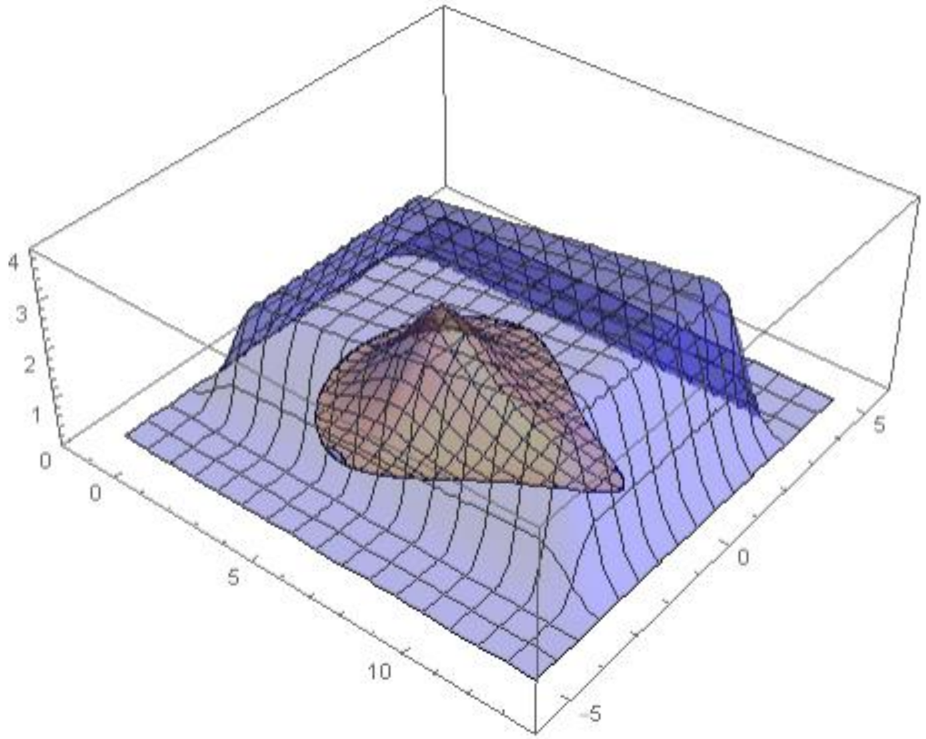
Let $\sigma \equiv 1$ on a neighborhood of K and...

$$\left(f^k, \sigma\pi_1^k, \dots, \sigma\pi_m^k\right) \to (f, \sigma\pi_1, \dots, \sigma\pi_m)$$

By assumption, $T\left(f^k, \sigma\pi_1^k, \dots, \sigma\pi_m^k\right) \to T(f, \sigma\pi_1, \dots, \sigma\pi_m)$

So the extension is continuous:

$$\bar{T}\left(f^k, \pi_1^k, \dots, \pi_m^k\right) \to \bar{T}(f, \pi_1, \dots, \pi_m)$$



Definition (Lang 2.3) Let $T \in \mathcal{D}_m(X)$ and $(u, v) \in \mathrm{Lip}_{\mathrm{loc}}(X) \times \left[\mathrm{Lip}_{\mathrm{loc}}(X)\right]^k$, with $0 \le k \le m$.

Define $T\lfloor(u, v)$ by the formula

$$(T\lfloor(u, v))(f, g) = T(uf, v, g), \qquad (f, g) \in \mathcal{D}^{m-k}(X)$$

$T\lfloor(u, v)$ is easily seen to be an $m - k$-current.

Proposition (Lang 2.4) Suppose $T \in \mathcal{D}_m(X)$, $m \geq 1$. $(f, \pi_1, \dots, \pi_m) \in \mathcal{D}^m(X)$. Then:

1. If $\pi_i = \pi_j$ for some $i \neq j$ then $T(f, \pi_1, \dots, \pi_m) = 0$.

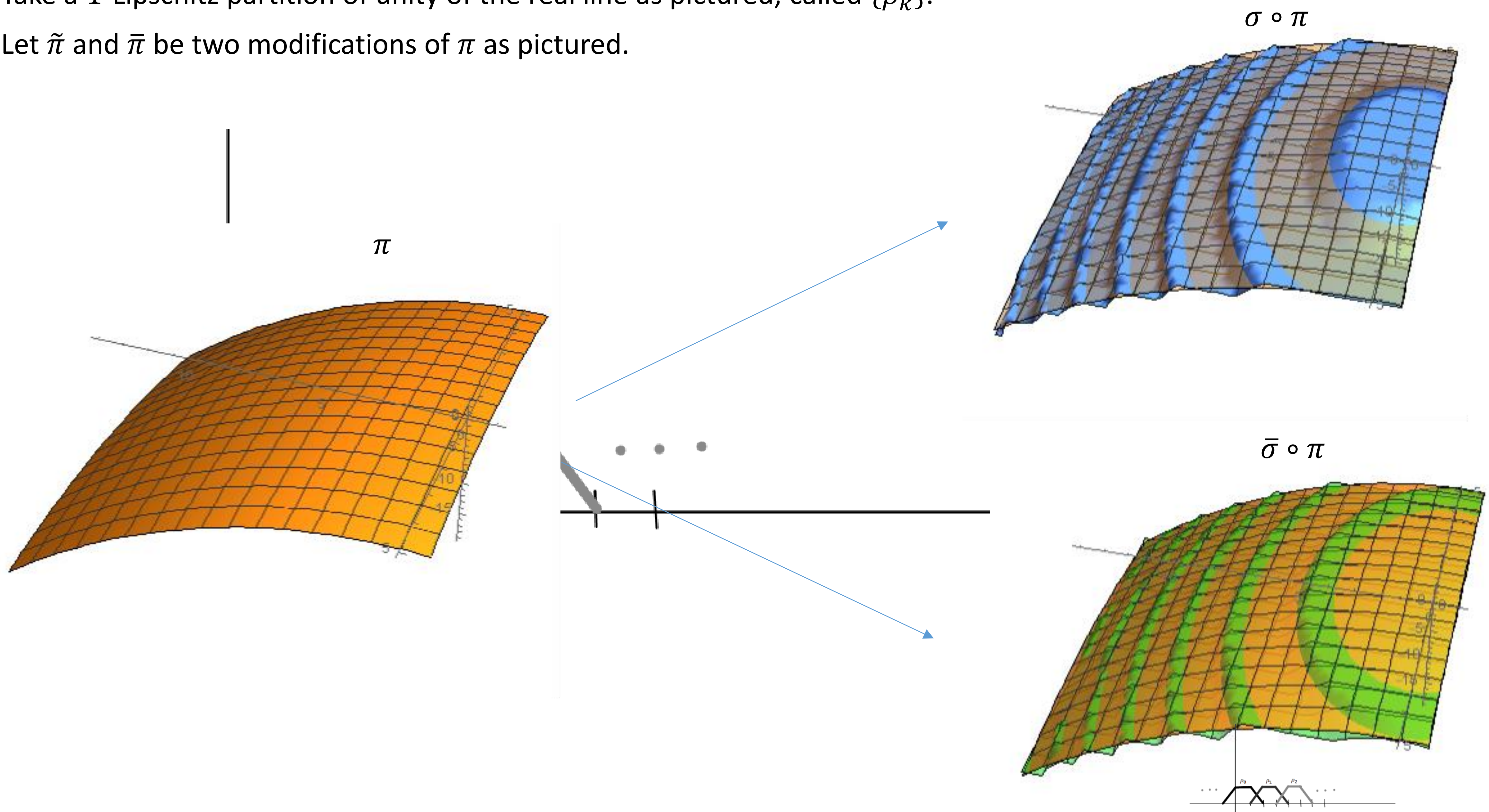
2. For $g, h \in \mathrm{Lip}_{\mathrm{loc}}(X)$,

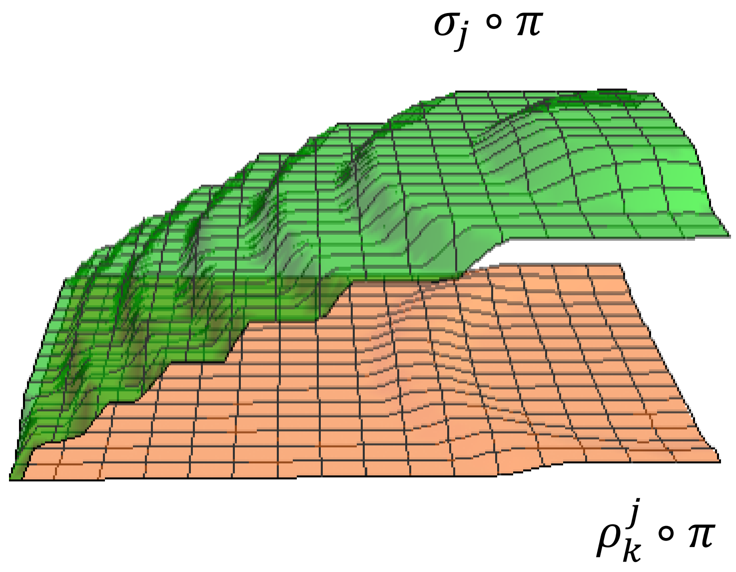
$$T(f, gh, \pi_2, \dots, \pi_m) = T(fg, h, \pi_2, \dots, \pi_m) + T(fh, g, \pi_2, \dots., \pi_m)$$

We first prove (1), the *alternating property*. Let us prove that $T(f,\pi,\pi)=0$ for $(f,\pi)\in\mathcal{D}^1(X)$.

Take a 1-Lipschitz partition of unity of the real line as pictured, called $\{\rho_k\}$.

Let $\tilde{\pi}$ and $\bar{\pi}$ be two modifications of π as pictured.

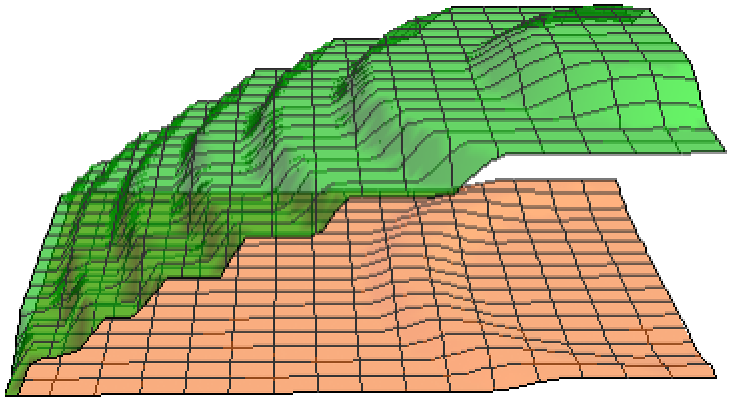



$\sigma_j \circ \pi$
$\rho_k^j \circ \pi$

Finite sum (why?)

$$T(f, \sigma_j \circ \pi, \bar{\sigma}_j \circ \pi) = \sum_{k \in \mathbb{Z}} T\left(\left(\rho_k^j \circ \pi\right) f, \sigma_j \circ \pi, \bar{\sigma}_j \circ \pi\right)$$

$$T(f, \pi, \pi) = 0, \quad \text{by strict locality}$$



Now we prove the *product rule* (2) $T(f, gh, \pi_2, \dots, \pi_m) = T(fg, h, \pi_2, \dots, \pi_m) + T(fh, g, \pi_2, \dots., \pi_m)$

It suffices to prove $T(f, g^2) = 2T(fg, g)$ (Why?)

$$T\left(f, (\sigma_j \circ g)(\bar{\sigma}_j \circ g)\right) = \sum_{k \in \mathbb{Z}} T\left(\left(\rho_k^j \circ g\right) f, (\sigma_j \circ g)(\bar{\sigma}_j \circ g)\right)$$

$$= \sum_{k \text{ even}} \frac{2k}{j} T\left(\left(\rho_k^j \circ g\right) f, \bar{\sigma}_j \circ g\right) + \sum_{k \text{ odd}} \frac{2k}{j} T\left(\left(\rho_k^j \circ g\right) f, \sigma_j \circ g\right)$$

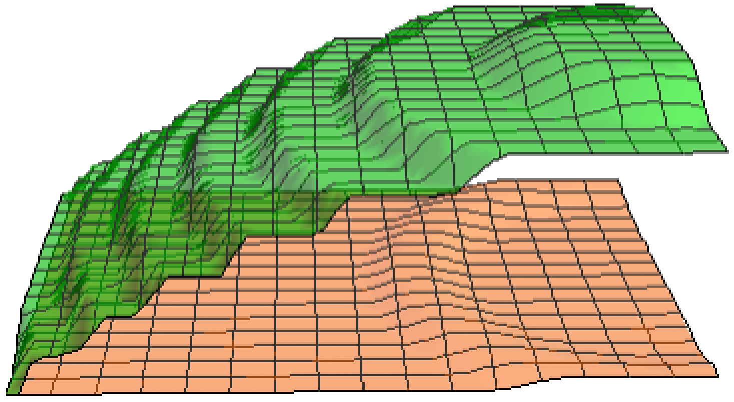
$$= \sum_{k \text{ even}} \frac{2k}{j} T\left(\left(\rho_k^j \circ g\right) f, \bar{\sigma}_j \circ g + \sigma_j \circ g\right) + \sum_{k \text{ odd}} \frac{2k}{j} T\left(\left(\rho_k^j \circ g\right) f, \sigma_j \circ g + \bar{\sigma}_j \circ g\right)$$

$$\tau_j = \sum_{k \in \mathbb{Z}} \frac{2k}{j} \rho_k^j$$

$$= T\left((\tau_j \circ g) f, (\sigma_j + \bar{\sigma}_j) \circ g\right)$$

$$\to T(gf, 2g)$$

■



Theorem (Chain Rule, Lang 2.5) $T(f, g\circ\pi) = T\big((g'\circ\pi)f, \pi\big)$ for 1-currents T and $g \in C^{1,1}(\mathbb{R})$

Proof

$$T(f, \pi^r) = T(r\pi^{r-1}f, \pi) \text{ by the product rule.}$$

Thus the chain rule holds for g a polynomial. Now suppose $g \in C^2(\mathbb{R})$

Invoke Stone-Weierstrass Theorem to find polynomials $p_j \to g$ in $C^2(\mathbb{R})$.

Finally, any $g \in C^{1,1}$ can be approximated by $g_j \in C^2(\mathbb{R})$ by convolution. ■

More generally:

Theorem (Chain Rule, Lang 2.5) Suppose $m, n \geq 1, T \in \mathcal{D}_m(X), U \subset \mathbb{R}^n$ open, $f \in \mathcal{D}(X)$ $\pi = (\pi_1, \dots, \pi_n) \in \mathrm{Lip}_{\mathrm{loc}}(X, U)$, $g = (g_1, \dots, g_m) \in [C^{1,1}(U)]^m$. If $n \geq m$ then

$$T(f, g \circ \pi) = \sum_{\lambda \in \Lambda(n,m)} T\left(f \det\left[\left(D_{\lambda(k)} g_i\right) \circ \pi\right]_{i,k=1}^{m}, \pi_{\lambda(1)}, \dots, \pi_{\lambda(m)}\right)$$

If $n < m$ then $T(f, g \circ \pi) = 0$

Proof Again, the theorem holds for polynomials, and follows from a density argument.

Proposition (Standard Example, Lang 2.6) Let $U \subset \mathbb{R}^m$ open, $m \geq 1$. Then every $u \in L^1_{\text{loc}}(U)$ induces a current $[u] \in \mathcal{D}_m(U)$ satisfying

$$[u](f, g) = \int_U uf \det(Dg)\ dx$$

Proof Locality and multilinearity are obvious. We prove continuity. Let $(f^j, g^j) \to (f, g) \in \mathcal{D}^m(U)$.

There exists $V \Subset U$ and $l > 0$ such that $\operatorname{spt}(f^j) \subset V$ and $\operatorname{Lip}(f^j) \leq l$ for all j, and $f^j \to f$ uniformly;

Moreover $\operatorname{Lip}\left(g_i^j|_V\right) \leq l$ for j, i and $g_i^j|_V \to g_i|_V$ uniformly. Put $h_i^j = g_i^j - g_i$ and we have

$$[u](f^j, g^j) - [u](f, g) = [u](f^j - f, g^j) + \sum_{i=1}^{m} [u]\left(f, g_1, \dots, g_{i-1}, h_i^j, g_{i+1}^j, \dots, g_m^j\right)$$

The first term tends to zero. Consider the summand $i = 1$. $uf \in L^1(V)$ so we need to show

$$\int_V v \det\left(D(h_1^j, g_2^j, \dots, g_m^j\right) dx \to 0, \qquad v \in L^1(V)$$

But $C_c^1(V) \subset L^1(V)$ is dense and the determinants are bounded in $L^\infty(V)$. So we can take $v \in C_c^1(V)$.

$$\int_V v \det\left(D\left(h_1^j, g_2^j, \dots, g_m^j\right)\right) dx = -\int_V h_1^j \det\left(D\left(v, g_2^j, \dots, g_m^j\right)\right) dx \qquad \text{(Stokes' Theorem)}$$

$\downarrow$

0

Definition (Support, Lang 3.1) Given $T \in \mathcal{D}_m(X)$, $m \geq 0$, its support $\mathrm{spt}(T)$ in X is the intersection of closed sets $C \subset X$ with the property that $T(f,\pi) = 0$ for $(f,\pi) \in \mathcal{D}^m(X)$ with $\mathrm{spt}(f) \cap C = \emptyset$.

$$\mathrm{spt}(T) = \bigcap\{C \text{ closed} : T(f,\pi) = 0 \text{ for } (f,\pi) \in \mathcal{D}^m(X) \text{ with } \mathrm{spt}(f) \cap C = \emptyset\}$$

$(*)$

Lemma (Support, Lang 3.2) Suppose $T \in \mathcal{D}_m(X)$, $m \geq 0$. Then:

(1) $\mathrm{spt}(T) = \{x \in X : (\varepsilon > 0)(\exists (f,\pi) \in \mathcal{D}^m(X))(\mathrm{spt}(f) \subset B(x,\varepsilon) \text{ and } T(f,\pi) \neq 0)\}$

(2) If $f|_{\mathrm{spt}(T)} = 0$ then $T(f,\pi_1,\dots,\pi_m) = 0$

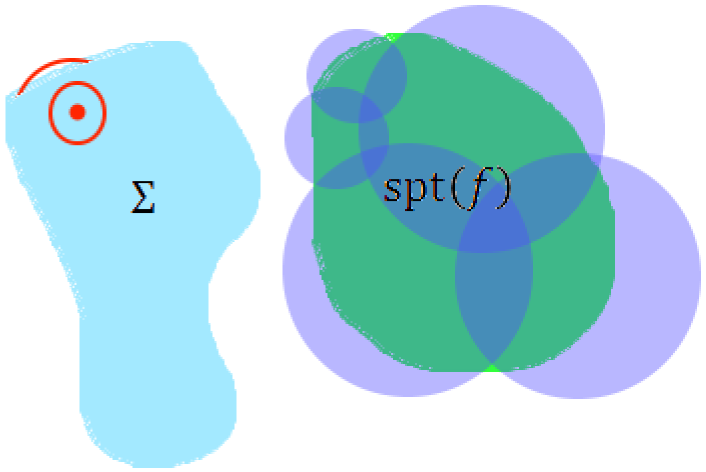
(3) $T(f,\pi_1,\dots,\pi_m) = 0$ if some π_i is constant on $\{f \neq 0\} \cap \mathrm{spt}(T)$.

Proof Let Σ be the set described in (1). Suppose $x \notin \mathrm{spt}(T)$. Let C be a closed set with property $(*)$ and $x \notin C$.

Let $\varepsilon > 0$ be such that $T(f,\pi) = 0$ whenever $\mathrm{spt}(f) \subset B(x,\varepsilon)$. Conclude $x \notin \Sigma$.

Let us now show that Σ has property $(*)$. This will show $\mathrm{spt}(T) \subset \Sigma$. Let $\mathrm{spt}(f) \cap \Sigma = \emptyset$.

Let $U_1,\dots,U_N$ be a covering of $\mathrm{spt}(f)$ by balls not touching Σ with property $(**)$



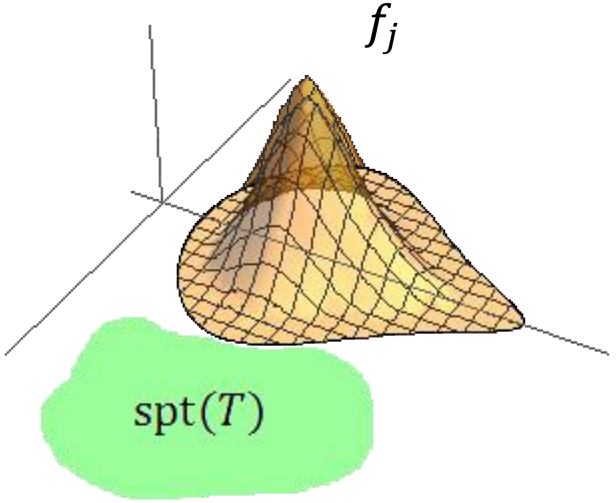
$$(\varepsilon > 0)(\exists (g,\pi) \in \mathcal{D}^m(X))(\mathrm{spt}(g) \subset B(x,\varepsilon) \text{ and } T(g,\pi) \neq 0)$$
$$(\exists \varepsilon > 0)(\forall (g,\pi) \in \mathcal{D}^m(X))(\mathrm{spt}(g) \subset B(x,\varepsilon) \Rightarrow T(g,\pi) = 0)$$

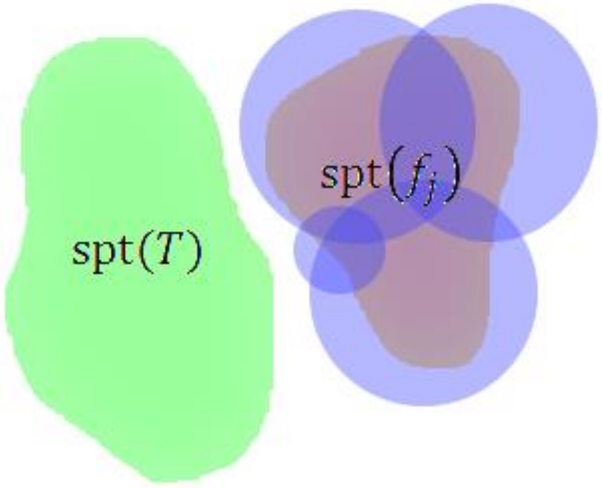
$(**)$

Decompose $f = \sum_{i=1}^N \varphi_i f$

$$T(f,\pi) = \sum_{i=1}^N T(\varphi_i f,\pi) = 0$$

Proof (continued) Now let us show that if $f|_{\mathrm{spt}(T)} = 0$ then $T(f, \pi_1, \dots, \pi_m) = 0$.





Each ball B has the property that
$$\mathrm{spt}(g) \subset B \Rightarrow T(g, \pi) = 0$$

Take a partition of unity subordinate to these balls and conclude $T(f_j, \pi) = 0$

By continuity, $T(f, \pi) = 0$.

Proof (continued) Finally we must show that $T(f, \pi_1, \ldots, \pi_m) = 0$ if some π_i is constant on $\{f \neq 0\} \cap \operatorname{spt}(T)$.

Assume WLG $m = 1$. Subtract a constant and assume $\pi = 0$ on $\{f \neq 0\} \cap \operatorname{spt}(T)$.

$$T(f, \pi) \longleftarrow T\big(f, \beta_j \circ \pi\big) - T\big(f(1-\sigma), \beta_j \circ \pi\big) = T\big(\sigma f, \beta_j \circ \pi\big) = 0$$

∎

Observe $\operatorname{spt}\big(\beta_j \circ \pi\big) \cap \operatorname{spt}\big(f|_{\operatorname{spt}(T)}\big) = \emptyset$.

Let $\operatorname{spt}\big(f|_{\operatorname{spt}(T)}\big) \prec \sigma \prec X \backslash \operatorname{spt}\big(\beta_j \circ \pi\big)$

0 because of part (2):
$g|_{\operatorname{spt}(T)} = 0 \Rightarrow T(g, \pi) = 0$

Proposition (Lang 3.3) Let $T \in \mathcal{D}_m(X)$, $A \subset X$ a locally compact subspace containing $\operatorname{spt}(T)$. Then there is a unique current $T_A \in \mathcal{D}_m(A)$ with the property that…

$$T_A(f, \pi_1, \dots, \pi_m) = T\left(\bar{f}, \bar{\pi}_1, \dots, \bar{\pi}_m\right)$$

… whenever $\bar{f}, \bar{\pi}_1, \dots, \bar{\pi}_\mathrm{m}$ are extensions of $f, \pi_1, \dots, \pi_m$ to all of X. Moreoever, $\operatorname{spt}(T_A) = \operatorname{spt}(T)$.

Proof Let $K \subset A$ be compact, $l \geq 0$ and $c > 0$. There exist $K \subset K' \subset X$, $l' \geq l$ and E an extension operator

$$E\colon \operatorname{Lip}_{K,l}(A) \cap \{\|f\|_\infty \leq c\} \to \operatorname{Lip}_{K',l'}(X)$$

E can be taken to be a MacShane extension times a cutoff function. If E and $\tilde{E}$ are two such extensions, then

$$T(Ef, E\pi_1, \dots, E\pi_m) - T\left(\tilde{E}f, \tilde{E}\pi_1, \dots, \tilde{E}\pi_m\right) = T\left(Ef - \tilde{E}f, E\pi_1, \dots, E\pi_m\right) + \sum_{i=1}^{m} T\left(\tilde{E}f, \tilde{E}\pi_1, \dots, \tilde{E}\pi_{i-1}, E\pi_i - \tilde{E}\pi_i, E\pi_{i+1}, \dots, E\pi_m\right)$$

Each of the terms vanishes by the previous lemma. So T_A is thus well-defined.
We used the fact that currents are determined by their values on $\mathcal{D}(X)^{m+1}$.

Definition (Boundary, Lang 3.4) The boundary of a current $T \in \mathcal{D}_m(X), m \geq 1$ is the current $\partial T \in \mathcal{D}_{m-1}(X)$ defined by

$$\partial T(f, \pi_1, \dots, \pi_{m-1}) \coloneqq T(\sigma, f, \pi_1, \dots, \pi_{m-1})$$

for $(f, \pi_1, \dots, \pi_{m-1}) \in \mathcal{D}^{m-1}(X)$, where $\sigma \in \mathcal{D}(X)$ is any function with $\sigma \equiv 1$ on $\{f \neq 0\} \cap \mathrm{spt}(T)$.

Lemma (Lang 3.5) $(\partial T)\lfloor(u, v) = T\lfloor(1, u, v) + (-1)^k \partial(T\lfloor(u, v))$

Proof

$$\begin{aligned}
\big((\partial T)\lfloor(u, v)\big)(f, g) &= \partial T(uf, v, g) \\
&= T(\sigma, uf, v, g) \\
&= T(\sigma f, u, v, g) + T(\sigma u, f, v, g) \\
&= T(f, u, v, g) + (-1)^k T(\sigma u, v, f, g) \\
&= (T\lfloor(1, u, v))(f, g) + (-1)^k (\partial(T\lfloor(u, v))(f, g)
\end{aligned}$$

Observe that if M is a manifold with boundary

$$[\![\partial M]\!](f dx_1 \wedge \cdots \wedge dx_{m-1}) = [\![M]\!](df \wedge dx_1 \wedge \cdots \wedge dx_{m-1})$$

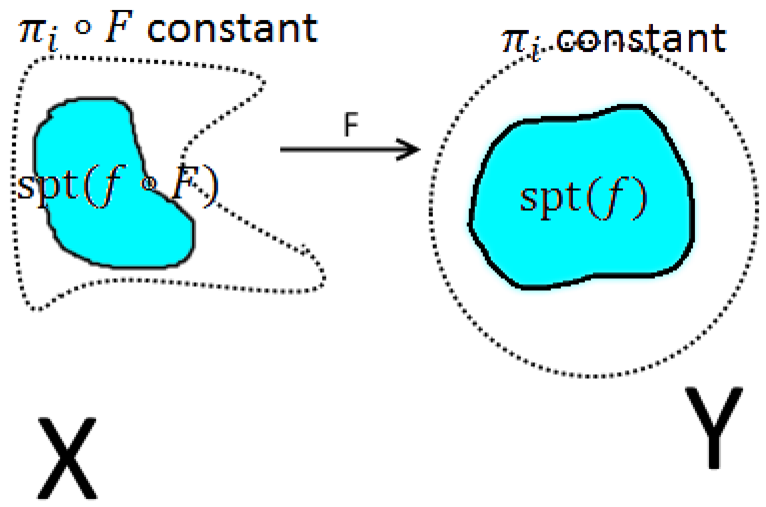
So this definition is simply meant to give us Stokes' Theorem.

Definition (Push-forward, Lang 3.6) Suppose $T \in \mathcal{D}_m(X)$, $A \subset X$ is a locally compact subspace containing $\mathrm{spt}(T)$. Suppose Y is another locally compact metric space. Suppose $F \in \mathrm{Lip}_{\mathrm{loc}}(A, Y)$ is proper. Define the pushforward:

$$F_\# T(f, \pi_1, \dots, \pi_m) \coloneqq T_A(f \circ F, \pi_1 \circ F, \dots, \pi_m \circ F)$$

For $(f, \pi_1, \dots, \pi_m) \in \mathcal{D}^m(Y)$.

Proof (that $F_\# T$ is a current): Multilinearity of $F_\# T$ follows immediately from multilinearity of T_A. The same is true for continuity. We prove locality. Suppose π_i is constant on a neighborhood of $\mathrm{spt}(f)$. Then $\pi_i \circ F$ is constant on a neighborhood of $\mathrm{spt}(f \circ F)$ in A.



By locality of T_A, $F_\# T(f, \pi_1, \dots, \pi_m) = T_A(f \circ F, \pi_1 \circ F, \dots, \pi_m \circ F) = 0$

Remark 1

$$\partial(F_\# T)(f,\pi) = (F_\# T)(\sigma, f, \pi) \qquad \sigma \equiv 1 \text{ on } \{f \neq 0\} \cap \operatorname{spt}(F_\# T)$$

$$= T_A(\sigma \circ F, f \circ F, \pi \circ F) \qquad \text{Note } \sigma \circ F \equiv 1 \text{ on } \{f \circ F \neq 0\} \cap \operatorname{spt}(T_A)$$

$$= \partial(T_A)(f \circ F, \pi \circ F)$$

$$= (\partial T)_A(f \circ F, \pi \circ F) \qquad \text{Easy lemma, omitted.}$$

$$= F_\#(\partial T)(f,\pi)$$

$$\boxed{\partial F_\# = F_\# \partial}$$

Remark 2

$$\boxed{(G \circ F)_\# = G_\# F_\#}$$

Lemma 3.7 Suppose $u \in L^1_{\text{loc}}(\mathbb{R}^m)$, $F \in \text{Lip}_{\text{loc}}(\mathbb{R}^m, \mathbb{R}^m)$, and $F|_{\text{spt}(u)}$ is proper. Then $F_\#[u] = [v]$ where $v \in L^1_{\text{loc}}(\mathbb{R}^m)$ satisfies

$$v(y) = \sum_{x \in F^{-1}\{y\}} u(x) \text{sgn } \det DF(x) \qquad \mathcal{L}^m\text{-a.e. } y \in \mathbb{R}^m$$

Proof Let $(f, \pi) \in \mathcal{D}^m(\mathbb{R}^m)$. Then…

$$F_\#[u](f,\pi) = \int_{\mathbb{R}^m} u(x) f\big(F(x)\big) \det D(\pi \circ F)_x \; dx$$

$$= \int_{\mathbb{R}^m} \underbrace{u(x) f\big(F(x)\big) \det D\pi_{F(x)} \text{ sgn} \det DF_x}_{h(x)} \, |\det DF_x| dx$$

$$= \int_{\mathbb{R}^m} \sum_{x \in F^{-1}\{y\}} h(x) \; dy \qquad \text{Area formula, c.f. Evans and Gariepy}$$

$$= \int_{\mathbb{R}^m} v(y) f(y) \det D\pi_y \; dy$$

$$= [v](f,\pi) \qquad \blacksquare$$

Definition (Mass, Lang 4.1) For $T \in \mathcal{D}_m(X)$, $V \subset X$ open, define the mass of T on V $M_V(T)$ as

$$M_V(T) = \sup_{*} \sum_{\lambda \in \Lambda} T\left(f_\lambda, \pi^\lambda\right)$$

$*$: Λ is a finite indexing set, $\left(f_\lambda, \pi^\lambda\right) \in \mathcal{D}(X) \times [\mathrm{Lip}_1(X)]^m$, $\mathrm{spt}(f_\lambda) \subset V$, $\sum_{\lambda \in \Lambda} |f_\lambda| \leq 1$.

Define $M(T) \coloneqq M_X(T)$ the total mass of T.

Denote $M_{m,\mathrm{loc}}(X)$ the vector space of $T \in \mathcal{D}_m(X)$ with $M_V(T) < \infty$ for $V \Subset X$.

Define $M_m(X) \coloneqq \{T \in \mathcal{D}_m(X) : M(T) < \infty\}$

Define $\|T\|(A) \coloneqq \inf\{M_V(T) \colon V \subset X \text{ open}, A \subset V\}$ for $T \in \mathcal{D}_m(X)$, $A \subset X$.

Mass is weak lower-semicontinuous, clearly. Mass is a norm on $M_m(X)$.

Proposition 4.2 $(M_m(X), M)$ is a Banach space.

Proof Sketch Given a Cauchy sequence $\{T_k\}_{k=1}^{\infty}$ in $(M_m(X), M)$, $\{T_k(f,\pi)\}_{k=1}^{\infty}$ is Cauchy for $(f,\pi) \in \mathcal{D}^m(X)$. One defines $T(f,\pi)$ to be the limit, then shows that it is a current and the limit of T_k. ■

Proposition 4.2 $(M_m(X), M)$ is a Banach space.

Proof Let $\{T_k\}_{k=1}^{\infty}$ be Cauchy in $(M_m(X), M)$. Let $\varepsilon > 0$. Let $(f, \pi) \in \mathcal{D}^m(X)$.

$$(T_k - T_l)(f, \pi_1, \dots, \pi_m) = \|f\|_\infty \prod_{i=1}^{m} \mathrm{Lip}\big(\pi_i|_{\mathrm{spt}(f)}\big)\, (T_k - T_l)\left(\frac{f}{\|f\|_\infty}, \frac{\pi_1}{\mathrm{Lip}\big(\pi_1|_{\mathrm{spt}(f)}\big)}, \dots, \frac{\pi_m}{\mathrm{Lip}\big(\pi_m|_{\mathrm{spt}(f)}\big)}\right)$$

$$\leq \|f\|_\infty \prod_{i=1}^{m} \mathrm{Lip}\big(\pi_i|_{\mathrm{spt}(f)}\big)\, M_m(T_k - T_l)$$

$$< \varepsilon, \qquad \text{for } k, l \text{ sufficiently large.}$$

Define $T(f, \pi) = \lim_{k \to \infty} T_k(f, \pi)$. T is $(m+1)$-multilinear and satisfies the locality condition.
For continuity: let $\big(f^j, \pi^j\big) \to (f, \pi)$ in $\mathcal{D}^m(X)$.

$$\big|T\big(f^j, \pi^j\big) - T(f, \pi)\big| \leq \big|T\big(f^j, \pi^j\big) - T_k\big(f^j, \pi^j\big)\big| + \big|T_k\big(f^j, \pi^j\big) - T_k(f, \pi)\big| + |T_k(f, \pi) - T(f, \pi)|$$

$$\leq 3\varepsilon, \qquad \text{for } j, k \text{ sufficiently large.}$$

Finally we must check that $M(T_k - T) \to 0$. We leave this as an easy exercise.

Remark (Mass for standard examples) Let $U \subset \mathbb{R}^m$ open, $T \in \mathcal{D}_m(U)$. Invoke the chain rule:

$$M_V(T) = \sup\left\{\sum_{\lambda\in\Lambda} T\left(f_\lambda, \pi_1^\lambda, \dots, \pi_m^\lambda\right) : \Lambda \text{ finite}, \sum_{\lambda\in\Lambda} |f_\lambda| \leq 1, \left(f_\lambda, \pi^\lambda\right) \in \mathcal{D}(U) \times [\mathrm{Lip}_1(U)]^m, \mathrm{spt}(f_\lambda) \subset V\right\}$$

$$= \sup\left\{\sum_{\lambda\in\Lambda} T\left(f_\lambda, \pi_1^\lambda, \dots, \pi_m^\lambda\right) : \Lambda \text{ finite}, \sum_{\lambda\in\Lambda} |f_\lambda| \leq 1, \left(f_\lambda, \pi^\lambda\right) \in \mathcal{D}(U) \times [\mathrm{Lip}_1(U) \cap C^{1,1}(U)]^m, \mathrm{spt}(f_\lambda) \subset V\right\}$$

$$= \sup\left\{\sum_{\lambda\in\Lambda} T\left(f_\lambda \det\left[\frac{\partial \pi_i^\lambda}{\partial x_k}\right]_{i,k=1}^m, \mathrm{Id}\right) : \Lambda \sum_{\lambda\in\Lambda} |f_\lambda| \leq 1, \left(f_\lambda, \pi^\lambda\right) \in \mathcal{D}(U) \times [\mathrm{Lip}_1(U) \cap C^{1,1}(U)]^m, \mathrm{spt}(f_\lambda) \subset V\right\}$$

$$= \sup\{T(f, \mathrm{Id}) : |f| \leq 1, \mathrm{spt}(f) \subset V\}$$

If $u \in L^1_{\mathrm{loc}}(U)$, we have

$$M_V([u]) = \int_V |u| dx$$

Theorem (Mass, Lang 4.3) Let $T \in \mathcal{D}_m(X)$.

(1) $\|T\|$ is a Borel regular measure.

(2) $\operatorname{spt}(\|T\|) = \operatorname{spt}(T)$ and $\|T\|\big(X \backslash \operatorname{spt}(T)\big) = 0$

(3) For open $V \subset X$,

$$\|T\|(V) = \sup_{\substack{K \subset X \text{ compact} \\ K \subset V}} \|T\|(K)$$

(4) If $T \in M_{m,\mathrm{loc}}(X)$ then $\|T\|$ is a Radon measure and

$$|T(f,\pi)| \leq \prod_{i=1}^{m} \operatorname{Lip}\big(\pi_i|_{\operatorname{spt}(f)}\big) \int_X |f| \, d\|T\|$$

Proof Recall the definitions

$$\|T\|(A) = \inf\{M_V(T) : V \subset X \text{ open}, A \subset V\}$$

$$M_V(T) = \sup_{*} \sum_{\lambda\in\Lambda} T\left(f_\lambda, \pi^\lambda\right)$$

We want to prove $\|T\|$ is a Borel regular measure. We begin by proving subadditivity for open sets $V \subset \bigcup_{i=1}^{\infty} V_i$

Let Λ and $\left(f_\lambda, \pi^\lambda\right)$ be as in the definition of $M_V(T)$, N the first index with $\bigcup_{i=1}^{N} V_i \supset K \coloneqq \bigcup_{\lambda\in\Lambda} \operatorname{spt}(f_\lambda)$.

Take a partition of unity on K, $\rho_1, \dots, \rho_N \in \mathcal{D}(X)$ subordinate to the V_i.

$$\sum_{\lambda\in\Lambda} T\left(f_\lambda, \pi^\lambda\right) = \sum_{\lambda\in\Lambda} \sum_{i=1}^{N} T\left(\rho_k f_\lambda, \pi^\lambda\right) = \sum_{i=1}^{N} \sum_{\lambda\in\Lambda} T\left(\rho_k f_\lambda, \pi^\lambda\right) \le \sum_{i=1}^{N} \|T\|(V_i)$$

$$\|T\|(V) \le \sum_{i=1}^{\infty} \|T\|(V_i)$$

Now subadditivity for arbitrary sets $A \subset \bigcup_{i=1}^{\infty} A_i$ follows (why?).

Also, $\|T\|$ satisfies Caratheodory's criterion: $\|T\|(A \cup B) = \|T\|(A) + \|T\|(B)$ whenever $d(A, B) > 0$. (Why?)

By Caratheodory's criterion, the Borel sets are $\|T\|$-measurable.

It is clear that $\|T\|$ is Borel regular: every $A \subset X$ is contained in a Borel set B of equal $\|T\|$-measure (why?).

We proved that $\|T\|$ is a Borel regular outer measure.

Proof (cont'd) Now we prove $\operatorname{spt}(\|T\|) = \operatorname{spt}(T)$ and that $\|T\|\big(X \backslash \operatorname{spt}(T)\big) = 0$.

Recall Lemma 3.2(1) $\operatorname{spt}(T) = \big\{x \in X : (\varepsilon > 0)\big(\exists (f, \pi) \in \mathcal{D}^m(X)\big)(\operatorname{spt}(f) \subset B(x, \varepsilon) \text{ and } T(f, \pi) \neq 0)\big\}$

And the definitions $\operatorname{spt}(\|T\|) = \{x \in X : (V \subset X \text{ open with } x \in V)(\|T\|(V) \neq 0)\}$

From these two characterizations, we easily have $\operatorname{spt}(T) \subset \operatorname{spt}(\|T\|)$

Next, if $x \notin \operatorname{spt}(T)$ then there is a closed set C with $x \notin C$ and $T(f, \pi) = 0$ for $\operatorname{spt}(f) \cap C = \emptyset$.

Let V open, $x \in V, V \cap C = \emptyset$. Then clearly $\|T\|(V) = 0$ so $x \notin \operatorname{spt}(\|T\|)$

We leave $\|T\|\big(X \backslash \operatorname{spt}(T)\big) = 0$ as an easy exercise.

Proof (cont'd) Now we prove (3): for open $V \subset X$, $\|T\|(V) = \sup\{\|T\|(K) : K \subset V \text{ compact}\}$.

Let $\alpha < \|T\|(V)$. Find Λ and $(f_\lambda, \pi^\lambda) \in \mathcal{D}(X) \times [\mathrm{Lip}_1(X)]^m$ such that $K = \bigcup_{\lambda \in \Lambda} \mathrm{spt}(f_\lambda) \subset V$, $\sum_\lambda |f_\lambda| \leq 1$ and

$$s \coloneqq \sum_{\lambda \in \Lambda} T(f_\lambda, \pi^\lambda) \geq \alpha$$

For U containing K, $\|T\|(U) \geq s \geq \alpha$, hence $\|T\|(K) \geq \alpha$.

This proves (3).

Proof (cont'd) We must prove for $T \in M_{m,\mathrm{loc}}(X)$ that $\|T\|$ is a Radon measure and

$$|T(f,\pi)| \leq \prod_{i=1}^{m} \mathrm{Lip}\left(\pi_i|_{\mathrm{spt}(f)}\right) \int_X |f| d\|T\|$$

$\|T\|$ is finite on compact sets so is a Radon measure. Now we prove the estimate. Consider $m = 0$ first.

Put $f_s = \min\{f, s\}$.

$$|T(f_t) - T(f_s)| = |T(f_t - f_s)| \leq \|T\|(\{f > s\})(t - s) \qquad \text{whenever } 0 \leq s < t$$

Hence $s \mapsto T(f_s)$ is a Lipschitz function with $|d/ds\, T(f_s)| \leq \|T\|(\{f > s\})$ for a.e. $s \geq 0$. Finally,

$$T(f) = T(f) - T(f_0) = \int_0^\infty (d/ds) T(f_s) ds$$

$$|T(f)| \leq \int_0^\infty \left|\frac{d}{ds} T(f_s)\right| ds \leq \int_0^\infty \|T\|(\{f > s\}) = \int_X f\, d\|T\|$$

Adjusting for $m \geq 1$ is easy, omitted.

∎

Theorem (Lang 4.4, Extended Functional) Let $T \in \mathbf{M}_{m,\mathrm{loc}}(X)$, $m \geq 0$. There is an extension $T: \mathcal{B}_c^\infty(X) \times \mathrm{Lip}_{\mathrm{loc}}(X)^m \to \mathbb{R}$ such that…

(1) Multilinearity

(2) continuity*

(3) locality

(4) Mass inequality

$$|T(f,\pi)| \leq \prod_{i=1}^{m} \mathrm{Lip}\left(\pi_i \big|_{\mathrm{spt}(f)}\right) \int_X |f| d\|T\|$$

Reason: $\mathcal{D}(X)$ is dense in $L^1(\|T\|) \supset \mathcal{B}_c^\infty(X)$

$f_j \to f$ if $\sup_j \|f_j\| < \infty$, $\bigcup_j \ \mathrm{spt}(f_j) \subset K$ some compact K, $f^j \to f$ pointwise on X

Lemma (Lang 4.6, Pushforwards and Mass) Suppose $T \in \mathbf{M}_{m,\text{loc}}(X)$, $m \geq 0$, Y locally compact metric space $F \in \text{Lip}_{\text{loc}}(X, Y)$, and $F|_{\text{spt}(T)}$ proper. Then $F_\# T \in \mathbf{M}_{m,\text{loc}}(Y)$ and

(1) For $(f, \pi) \in \mathcal{B}_c^\infty(Y) \times \left[\text{Lip}_{\text{loc}}(Y)\right]^m$ and $\sigma \in \mathcal{B}_c^\infty$ with $\sigma = 1$ on $\{f \circ F \neq 0\} \cap \text{spt}(T)$,

$$F_\# T(f, \pi) = T(\sigma(f \circ F), \pi \circ F)$$

(2) For Borel $B \subset Y$,

$$\mathbf{M}_V\left(F_\# T\Big|_B\right) \leq \text{Lip}\left(F\Big|_{F^{-1}(B) \cap \text{spt}(T)}\right)^m \|T\|(F^{-1}(V))$$

Proof Suppose $T \in \mathbf{M}_{m,\mathrm{loc}}(X)$, $m \geq 0$, Y locally compact metric space

$F \in \mathrm{Lip}_{\mathrm{loc}}(X, Y)$, and $F|_{\mathrm{spt}(T)}$ proper. We need to show $F_\# T \in \mathbf{M}_{m,\mathrm{loc}}(Y)$

Observe:

$$\mathbf{M}_V(F_\# T) = \sup_{*} \sum_{\lambda \in \Lambda} F_\# T\left(f_\lambda, \pi^\lambda\right) \qquad \sigma \in \mathcal{D}(X), \qquad \sigma \equiv 1 \text{ on } F^{-1}(V) \cap \mathrm{spt}(T)$$

$$= \sup_{*} \sum_{\lambda \in \Lambda} T\left(\sigma(f_\lambda \circ F), \pi^\lambda \circ F\right)$$

$$\leq \left(\mathrm{Lip}\left(F\Big|_{\mathrm{spt}(\sigma)}\right)\right)^m \|T\|(V)$$

This proves (2) and in particular that $F_\# T \in \mathbf{M}_{m,\mathrm{loc}}(Y)$

$*$: Λ finite, $\left(f_\lambda, \pi^\lambda\right) \in \mathcal{D}(X) \times [\mathrm{Lip}_1(X)]^m$, $\sum_\lambda |f_\lambda| \leq 1$, $\mathrm{spt}(f_\lambda) \subset V$

Let us now show: (1) For $(f,\pi) \in \mathcal{B}_c^\infty(Y) \times \left[\mathrm{Lip}_{\mathrm{loc}}(Y)\right]^m$ and $\sigma \in \mathcal{B}_c^\infty$ with $\sigma = 1$ on $\{f \circ F \neq 0\} \cap \mathrm{spt}(T)$,

$$F_\# T(f,\pi) = T(\sigma(f \circ F), \pi \circ F)$$

Indeed, this is true for $\sigma \in \mathcal{D}(X)$ with $\sigma = 1$ on $\{f \circ F \neq 0\} \cap \mathrm{spt}(T)$.

Now if $\sigma \in \mathcal{B}_c^\infty$, we can approximate σ by $\tau \in \mathcal{D}(X)$ and take a limit to prove the statement.

Now let us show (2): For Borel $B \subset Y$, $\mathbf{M}_V\left(F_\# T\Big|_B\right) \leq \mathrm{Lip}\left(F\Big|_{F^{-1}(B)\cap \mathrm{spt}(T)}\right)^m \|T\|(F^{-1}(V))$

Take $(f,\pi) \in \mathcal{D}(X) \times [\mathrm{Lip}_1(X)]^m$, $\sigma = \chi_{F^{-1}(B)\cap\{f\circ F\neq 0\}}$. Then,

$$\begin{aligned}\left(F_\# T\Big|_B\right)(f,\pi) &= F_\# T(\chi_B f, \pi) \\ &= T(\sigma(f\circ F), \pi \circ F) \\ &\leq \left(\mathrm{Lip}\, F\Big|_{\mathrm{spt}(\sigma)}\right)^m \int_{F^{-1}(B)} |f \circ F| d\|T\|\end{aligned}$$

Lemma (Lang 4.7, Characterizing $\|T\|$) Suppose $T \in \mathbf{M}_{m,\mathrm{loc}}(X)$, $B \subset X$ is σ-finite with respect to $\|T\|$ or open. Then:

$$\|T\|(B) = \sup_{*} \sum_{\lambda \in \Lambda} T(f_\lambda, \pi^\lambda)$$

Moreover, $\|T|B\| = \|T\||_B$

Proof Recall 4.4(4)

$$|T(f,\pi)| \leq \prod_{i=1}^{m} \mathrm{Lip}\left(\pi_i\Big|_{\mathrm{spt}(f)}\right) \int_X |f| d\|T\|$$

Thus

$$\sum_{\lambda \in \Lambda} T(f_\lambda, \pi^\lambda) \leq \|T\|(B)$$

On the other hand, let $\varepsilon > 0$. Let V open contain B with $\|T\|(V \setminus \mathrm{B}) \leq \varepsilon$. Choose $\alpha < \|T\|(V)$ and find (f_λ, π^λ) satisfying $(*)$ with

$$\alpha \leq \sum_{\lambda \in \Lambda} T(f_\lambda, \pi^\lambda) = \sum_{\lambda \in \Lambda} T(\chi_B f_\lambda, \pi^\lambda) + \sum_{\lambda \in \Lambda} T(\chi_{V\setminus \mathrm{B}} f_\lambda, \pi^\lambda)$$

$$\left|\sum_{\lambda \in \Lambda} T(\chi_{V\setminus \mathrm{B}} f_\lambda, \pi^\lambda)\right| \leq \varepsilon$$

$$\sum_{\lambda \in \Lambda} T(\chi_B f_\lambda, \pi^\lambda) \geq \alpha - \varepsilon$$

$*$: Λ finite, $(f_\lambda, \pi^\lambda) \in \mathcal{B}_c^\infty \times [\mathrm{Lip}_1(X)]^m$, $\sum_\lambda |f_\lambda| \leq \chi_B$

We've proved that

$$\|T\|(B) = \sup_{*} \sum_{\lambda\in\Lambda} T\left(f_\lambda, \pi^\lambda\right)$$

Now we must prove that $\|T\||_B = \|T|B\|$.

Choose A borel.

$$\|T\|\Big|_B (A) = \|T\|(A\cap B) = \sup_{**} \sum_{\lambda\in\Lambda} T\left(f_\lambda, \pi^\lambda\right) = \sup_{***} \sum_{\lambda\in\Lambda} T\left(f_\lambda \chi_B, \pi^\lambda\right) = \|T|B\|(A)$$

$\blacksquare$

$**$: Λ finite, $\left(f_\lambda, \pi^\lambda\right) \in \mathcal{B}_c^\infty(X) \times [\mathrm{Lip}_1(X)]^m$, $\sum_\lambda |f_\lambda| \leq \chi_{B\cap A}$

$***$: Λ finite, $\left(f_\lambda, \pi^\lambda\right) \in \mathcal{B}_c^\infty(X) \times [\mathrm{Lip}_1(X)]^m$, $\sum_\lambda |f_\lambda| \leq \chi_A$

$$N_V(T) = M_V(T) + M_V(\partial T)$$

$$N_{m,\mathrm{loc}}(X) = \{T \in \mathcal{D}_m(X) : N_V(T) < \infty \text{ for } V \Subset X\}$$

$$N(T) = N_X(T)$$

$$N_m(X) = \{T \in \mathcal{D}_m(X) : N(T) < \infty\}$$

Proposition $N_m(X)$ is a Banach space.

Proof If $\{T_i\}$ in $N_m(X)$ is Cauchy, then $\{T_i\}$ and $\{\partial T_i\}$ are Cauchy in $M_m(X)$ and $M_{m-1}(X)$ respectively.

So they have limits T^* and ∂T^* in $M_m(X)$ and $M_{m-1}(X)$.

$T_i \to T^*$ in $N_m(X)$, proving completeness.

■

Observation If $T \in N_{m,\mathrm{loc}}(X)$ and $(u, v) \in \mathrm{Lip}_{\mathrm{loc}}(X) \times \left[\mathrm{Lip}_{\mathrm{loc}}(X)\right]^k$, then $\partial(T|(u,v)) = (-1)^k(\partial T|(u,v)) - T|(1,u,v)$.

Hence

$$M_V\big(\partial(T|(u,v))\big) \le \prod_{i=1}^{m}\big(\mathrm{Lip}(v_i|V)\big)\int_V u\, d\|\partial T\| + \prod_{i=1}^{m}\big(\mathrm{Lip}(v_i|V)\big)\,\mathrm{Lip}(u|V)\|T\|(V)$$

So $T|(u,v) \in N_{m,\mathrm{loc}}(X)$

Observation Pushforwards of locally normal currents are locally normal.

Lemma (Lang 5.2, Uniform Continuity of Locally Normal Currents) Let $T \in N_{m,\mathrm{loc}}(X)$. Then,

(1) For $(f, g_1, g_2, \dots, g_m) \in \mathcal{D}(X) \times \mathrm{Lip}_{\mathrm{loc}}(X) \times [\mathrm{Lip}_1(X)]^{m-1}$,

$$|T(f,g)| \leq \mathrm{Lip}(f) \int_{\mathrm{spt}(f)} |g_1| d\|T\| + \int_X |f g_1| d\|\partial T\|$$

(2) For $(f,g), (\tilde{f}, \tilde{g}) \in \mathcal{D}(X) \times [\mathrm{Lip}_1(X)]^m$,

$$\left|T(f,g) - T(\tilde{f}, \tilde{g})\right| \leq \int_X \left|f - \tilde{f}\right| d\|T\| + \sum_{i=1}^{m} \mathrm{Lip}(f) \int_{\mathrm{spt}(f)} |g_i - \tilde{g}_i| d\|T\| + \sum_{i=1}^{m} \int_X |f||g_i - \tilde{g}_i| d\|\partial T\|$$

Proof Omitted; not interesting.

Lemma (Lang 5.3, Convergence Criterion) Suppose X is compact, $\mathcal{F} \subset \mathrm{Lip}_1(X)$ is dense in supremum norm $\|\cdot\|_\infty$.

Suppose (T_n) is a bounded sequence in $N_m(X)$, $m \geq 0$, with $M = \sup_n N(T_n) < \infty$.

Suppose further that $T_n(f, g)$ has a limit, which we'll denote $T(f, g)$, for $f, g \in \mathcal{F} \times \mathcal{F}^m$.

Then T_n converges weakly to a $T \in N_m(X)$.

Proof idea We must show that the natural limit $T(f, g) = \lim T_n(f, g)$ extends from $\mathcal{F} \times \mathcal{F}^m$ to $\mathcal{D}^m(X)$.

So we need local uniform continuity. Use the uniform continuity estimate…

$$\left|T(f,g) - T(\tilde{f}, \tilde{g})\right| \leq \int_X |f - \tilde{f}| d\|T\| + \sum_{i=1}^{m} \mathrm{Lip}(f) \int_{\mathrm{spt}(f)} |g_i - \tilde{g}_i| d\|T\| + \sum_{i=1}^{m} \int_X |f||g_i - \tilde{g}_i| d\|\partial T\|$$

Theorem (Lang 5.4, Compactness) Suppose (T_n) is a sequence in $N_{m,\mathrm{loc}}(X)$, $m \geq 0$, with $\mathrm{spt}(T_n)$ separable, Suppose also $\sup_n N_V(T_n) < \infty$, for open $V \Subset X$.

Then some subsequence converges weakly to a $T \in N_{m,\mathrm{loc}}(X)$

Proof Assume first X compact, so we can take a countable dense $\mathcal{F} \subset \mathrm{Lip}_1(X)$. A diagonalization argument yields that a subsequence T_{n_k} converges for $(f, g) \in \mathcal{F} \times \mathcal{F}^m$.

Integer Rectifiable Currents We say $T \in \mathcal{D}_m(X)$ is a locally *integer rectifiable current* if:

1. $T \in \mathbf{M}_{m,\mathrm{loc}}(X)$
2. Whenever $B \Subset X$ is Borel and $\pi \in \mathrm{Lip}(X, \mathbb{R}^m)$, we have $\pi_\#(T|B) = [u]$ for some $u \in L^1(\mathbb{R}^m, \mathbb{Z})$
3. $\|T\|$ is concentrated on a countably $\mathcal{H}^m$-rectifiable Borel set $B \subset X$.

Denote the set of such currents $\mathcal{I}_{m,\mathrm{loc}}(X)$. Define $\mathcal{I}_m(X) = \mathcal{I}_{m,\mathrm{loc}}(X) \cap \mathbf{M}_m(X)$

Facts about Integer Rectifiable Currents

1. *Parametric Representation*: $T \in \mathcal{I}_{m,\mathrm{loc}}(X)$ if and only if

$$T = \sum_{i=1}^{\infty} F_{i\#}[u_i], \qquad u_i \in L^1(\mathbb{R}^m, \mathbb{Z}), \qquad F_i\colon \mathbb{R}^m \to X \text{ bi-Lipschitz}, \quad \|T\|(A) = \sum_{i=1}^{\infty} \|T_i(A)\|$$

2. $\mathcal{I}_{m,\mathrm{loc}}(X) \cap \mathbf{N}_{m,\mathrm{loc}}(X)$ is locally compact.

Part II: an Application to the Heisenberg Group

Theorem (Zust, 1.3) Let X be a quasiconvex compact metric space with $\pi_1^{\mathrm{Lip}}(X) = 0$, and let $\varphi: (X, d_X) \to (\mathbb{H}, d_{cc})$ be Holder continuous of order $\alpha > 2/3$.
Then φ factors through a tree.

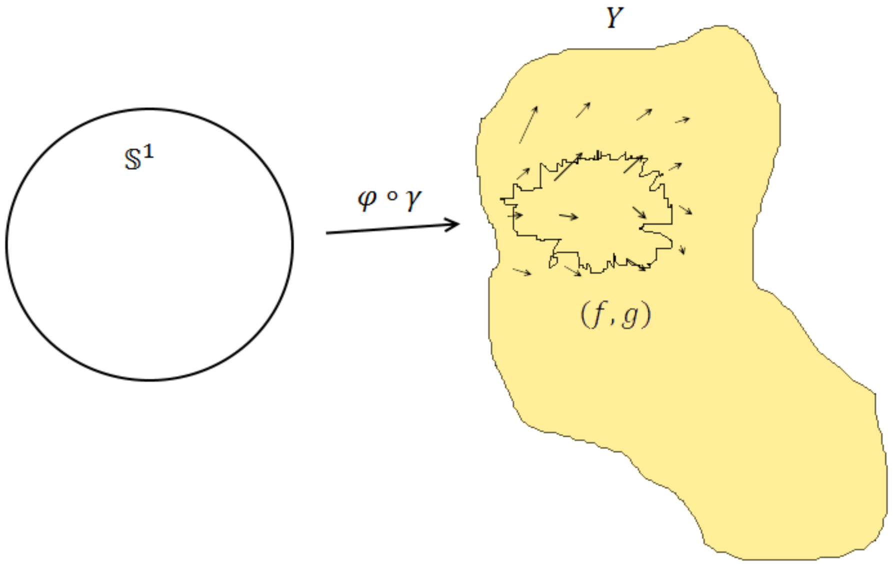
Proof Outline

Definition We say $\varphi: X \to Y$ has property (T) if for $x, x' \in X$ with $\varphi(x) \neq \varphi(x')$ there exists a point $y \in Y$ such that $\varphi \circ \gamma$ passes through y for all curves $\gamma: x \rightsquigarrow x'$.
Theorem (Zust 1.1) If X is C-quasiconvex compact with $H_1(X) = 0$ or $H_1^{\mathrm{Lip}}(X) = 0$ and $\varphi: X \to Y$ is σ-continuous with property (T), then φ factors through a tree + estimates and contractibility
Proposition (Zust 4.1) Let X be quasiconvex compact, $H_1(X) = 0$ or $H_1^{\mathrm{Lip}}(X) = 0$, $\varphi: X \to Y$ Holder continuous of order $\alpha > 1/2$, and suppose $(\varphi \circ \gamma)_\# [\![\mathbb{S}^1]\!] = 0$ for closed Lipschitz curves $\gamma: \mathbb{S}^1 \to X$. Then φ has property (T).
Lemma (Zust 4.6) Let $Q \subset \mathbb{R}^2$ be a square and $\varphi: Q \to \mathbb{H}$ Holder continuous of order $\alpha > \frac{2}{3}$. Then the pushforward $\varphi_\# [\![Q]\!] = 0$ viewed as a current in $\mathbb{R}^3$.

Notice, we have implicitly assumed:

If $\varphi: X \to Y$ is $\alpha > 1/2$ Holder continuous and $\gamma: \mathbb{S}^1 \to X$ a Lipschitz curve, then $(\varphi \circ \gamma)_\# [\![\mathbb{S}^1]\!]$ is a well-defined 1-current.

This can be done in several conceptually different ways.



We need to make sense of the expression

$$(\varphi \circ \gamma)_\# [\![\mathbb{S}^1]\!](f, g) = [\![\mathbb{S}^1]\!](f \circ \varphi \circ \gamma, g \circ \varphi \circ \gamma) = \int_{\mathbb{S}^1} (f \circ \varphi \circ \gamma)\, d(g \circ \varphi \circ \gamma) = \int_{\mathbb{S}^1} \bar{f}\, d\bar{g}$$

Riemann-Stieltjes Integration

Mollification

Sobolev Extension

$$= \mathrm{RS} \int_0^1 \bar{f}\, d\bar{g}$$

$$= \lim_{\varepsilon \to 0} \int_0^1 \bar{f}_\varepsilon\, d\bar{g}_\varepsilon$$

$$= \lim_{\varepsilon \to 0} \int_{\mathbb{B}^2} d\bar{\bar{f}} \wedge d\bar{\bar{g}}$$

All three give the same number, so take your pick.

Theorem (Zust, 1.3) Let X be a quasiconvex compact metric space with $\pi_1^{\text{Lip}}(X) = 0$, and let $\varphi: (X, d_X) \to (\mathbb{H}, d_{cc})$ be Holder continuous of order $\alpha > 2/3$. Then φ factors through a tree.

Proof Outline

Definition We say $\varphi: X \to Y$ has property (T) if for $x, x' \in X$ with $\varphi(x) \neq \varphi(x')$ there exists a point $y \in Y$ such that $\varphi \circ \gamma$ passes through y for all curves $\gamma: x \rightsquigarrow x'$.
Theorem (Zust 1.1) If X is C-quasiconvex compact with $H_1(X) = 0$ or $H_1^{\text{Lip}}(X) = 0$ and $\varphi: X \to Y$ is σ-continuous with property (T), then φ factors through a tree (+ estimates and contractibility)
Proposition (Zust 4.1) Let X be quasiconvex compact, $H_1(X) = 0$ or $H_1^{\text{Lip}}(X) = 0$, $\varphi: X \to Y$ Holder continuous of order $\alpha > 1/2$, and suppose $(\varphi \circ \gamma)_\# [\![\mathbb{S}^1]\!] = 0$ for closed Lipschitz curves $\gamma: \mathbb{S}^1 \to X$. Then φ has property (T).
Lemma (Zust 4.6) Let $Q \subset \mathbb{R}^2$ be a square and $\varphi: Q \to \mathbb{H}$ Holder continuous of order $\alpha > \frac{2}{3}$. Then the pushforward $\varphi_\# [\![Q]\!] = 0$ viewed as a current in $\mathbb{R}^3$.

Proof of 4.1: If $\varphi: X \to Y$ $\alpha > \frac{1}{2}$ Holder continuous, pushes forward Lipschitz loops to zero currents, then φ has property (T).

Proof Fix $x, x' \in X$ with $\varphi(x) \neq \varphi(x')$. Let $\mu, \mu': x \rightsquigarrow x'$ Lipschitz. Now $(\varphi \circ \mu)_\# [\![0,1]\!], (\varphi \circ \mu')_\# \in \mathcal{D}_1(Y)$.

They are non-zero currents since they have non-zero boundary.

But a 1-current cannot have a support consisting of finitely many points.

So there is a $y \in Y$ not equal to $\varphi(x)$ or $\varphi(x')$, belonging to the support $\operatorname{spt}\big((\varphi \circ \mu)_\# [\![0,1]\!]\big)$

Clearly y must be in the image of $\varphi \circ \mu$.

i.e. $(\varphi \circ \gamma)_\# (\mathbb{S}^1) = 0$

Let $\gamma = \mu * \mu'^{-1}: \mathbb{S}^1 \to X$. $0 = (\varphi \circ \gamma)_\# [\![\mathbb{S}^1]\!] = (\varphi \circ \mu)_\# [\![0,1]\!] - (\varphi \circ \mu')_\# [\![0,1]\!]$

Thus $(\varphi \circ \mu)_\# [\![0,1]\!] = (\varphi \circ \mu')_\# [\![0,1]\!]$, and so $y \in \operatorname{spt}\big((\varphi \circ \mu)_\# [\![0,1]\!]\big) = \operatorname{spt}\big((\varphi \circ \mu')_\# [\![0,1]\!]\big)$

So y is also in the image of $\varphi \circ \mu'$

This is property (T) ∎

Theorem (Zust, 1.3) Let X be a quasiconvex compact metric space with $\pi_1^{\mathrm{Lip}}(X) = 0$,
and let $\varphi: (X, d_X) \to (\mathbb{H}, d_{cc})$ be Holder continuous of order $\alpha > 2/3$.
Then φ factors through a tree.

Proof Outline

Definition We say $\varphi: X \to Y$ has property (T) if for $x, x' \in X$ with $\varphi(x) \neq \varphi(x')$ there exists a point $y \in Y$ such that $\varphi \circ \gamma$ passes through y for all curves $\gamma: x \rightsquigarrow x'$.
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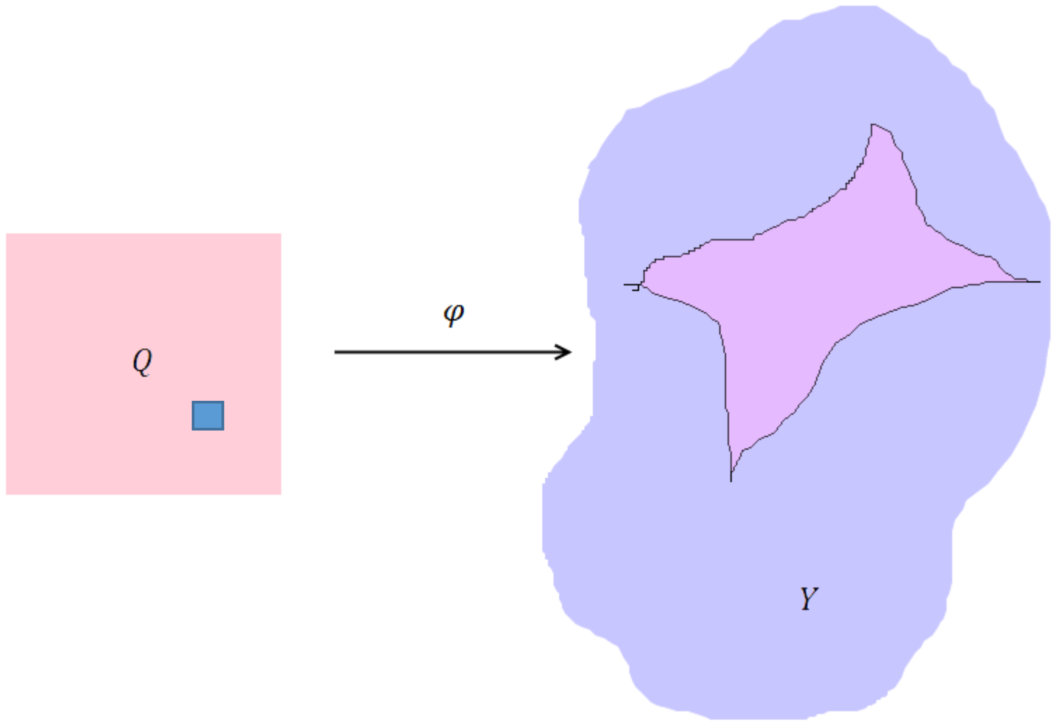
Again we need to check that we have a well-defined current $\varphi_\# [\![Q]\!]$ before proceeding to prove the lemma.



We need to make sense of $\varphi_\# [\![Q]\!](f, g_1, g_2)$

$$\varphi_\# [\![Q]\!](f, g_1, g_2) = [\![Q]\!](f \circ \varphi, g_1 \circ \varphi, g_2 \circ \varphi)$$

$$= \int_Q \bar{f}\, d\bar{g}_1 \wedge d\bar{g}_2$$

Again, we have options.

$$= \lim_{\varepsilon \to 0} \int_Q \bar{f}_\varepsilon\, d\bar{g}_{1\varepsilon} \wedge d\bar{g}_{2\varepsilon}$$

$$= \mathrm{Z} \int_Q \bar{f}\, d\bar{g}_1 \wedge d\bar{g}_2 = \lim_{n \to \infty} \sum_{Q_i \in \mathcal{P}_n(Q)} \bar{f}(b_{Q_i}) \int_{\partial Q_i} \bar{g}_1 d\bar{g}_2$$

$$= \int_{Q \times (0,1)} d\bar{\bar{f}} \wedge d\bar{\bar{g}}_1 \wedge d\bar{\bar{g}}_2$$

Theorem (Zust, 1.3) Let X be a quasiconvex compact metric space with $\pi_1^{\mathrm{Lip}}(X) = 0$, and let $\varphi\colon (X, d_X) \to (\mathbb{H}, d_{cc})$ be Holder continuous of order $\alpha > 2/3$.
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Proof of Lemma (Zust 4.6): Let $Q \subset \mathbb{R}^2$ be a square, $\varphi: Q \to \mathbb{H}$ $\alpha > \frac{2}{3}$ Holder continuous. Then $\varphi_{\#}[\![Q]\!] = 0$.

Proof First recall that an $\alpha > \frac{1}{2}$ Holder continuous curve $\gamma: [a, b] \to \mathbb{H}$ is weakly horizontal in the sense that

$$\int_a^b d\gamma_t + 2(\gamma_y d\gamma_x - \gamma_x d\gamma_y) = 0$$

In fact, more can be said: if $f: [a, b] \to \mathbb{R}$ is $\alpha > \frac{1}{2}$ Holder continuous, then

$$\int_a^b f\,\left(d\gamma_t + 2(\gamma_y d\gamma_x - \gamma_x d\gamma_y)\right) = 0$$

Let $f = \gamma_x$ and assume now that γ is a closed curve.

$$\int_a^b \gamma_x d\gamma_t = \int_a^b 2\gamma_x^2 d\gamma_y - \int_a^b 2\gamma_y\gamma_x d\gamma_x = \int_a^b 2\gamma_x^2 d\gamma_y + \int_a^b \gamma_x^2 d\gamma_y \;= 3\int_a^b \gamma_x^2 d\gamma_y$$

Similarly

$$\int_a^b \gamma_y d\gamma_t = -3\int_a^b \gamma_y^2 d\gamma_x$$

Proof of Lemma (Zust 4.6): Let $Q \subset \mathbb{R}^2$ be a square, $\varphi: Q \to \mathbb{H}$ $\alpha > \frac{2}{3}$ Holder continuous. Then $\varphi_\# [\![Q]\!] = 0$.

Proof We proved

$$(*) \quad \int_a^b \gamma_x d\gamma_z = 3 \int_a^b \gamma_x^2 d\gamma_y \qquad \qquad \int_a^b \gamma_y d\gamma_z = -3 \int_a^b \gamma_y^2 d\gamma_x$$

With these we compute, for $\omega_1, \omega_2, \omega_3$ Lipschitz

$$\varphi_\# [\![Q]\!](\omega_1 dy \wedge dt + \omega_2 dx \wedge dt + \omega_3 dx \wedge dy) = [\![Q]\!](\varphi^*(\omega_1 dy \wedge dt + \omega_2 dx \wedge dt + \omega_3 dx \wedge dy))$$

$$= [\![Q]\!]\left(\bar{\omega}\, d\varphi_x \wedge d\varphi_y\right)$$

This is correct by $(*)$, but requires more justification

$$= \lim_{n\to\infty} \sum_{Q_i \in \mathcal{P}_n(Q)} \bar{\omega}\left(b_{Q_i}\right) \int_{\partial Q_i} \varphi_x \, d\varphi_y$$

$$= \lim_{n\to\infty} \sum_{Q_i \in \mathcal{P}_n(Q)} \bar{\omega}\left(b_{Q_i}\right) \int_{\partial Q_i} \frac{1}{2}\left(\varphi_x d\varphi_y - \varphi_y d\varphi_x\right)$$

$$= 0$$

Proof of Lemma (Zust 4.6): Let $Q \subset \mathbb{R}^2$ be a square, $\varphi: Q \to \mathbb{H}$, $\beta > \frac{2}{3}$ Holder continuous. Then $\varphi_\# [\![Q]\!] = 0$.

Alternative Proof Let $\alpha = dt + 2(ydx - xdy)$ be the contact form for $\mathbb{H}$ with $\ker(\alpha) = H\mathbb{H}$.

Obvious estimates with convolutions, using the Holder continuity of φ and the Koranyi metric yield

$$\|\varphi_\varepsilon^* \alpha\|_\infty < C\varepsilon^{2\gamma-1}$$

And also for arbitrary 1-forms κ on $\mathbb{R}^3 = \mathbb{H}$ we have

$$\|\varphi_\varepsilon^* \kappa\|_\infty < C\varepsilon^{\gamma-1}$$

Observe that we have $dx \wedge dy = \frac{1}{4} d\alpha$, $dx \wedge dt = dx \wedge \alpha - \frac{x}{2} d\alpha$, and $dy \wedge dt = dy \wedge \alpha + \frac{y}{2} d\alpha$

Thus,

$$\varphi_\# [\![Q]\!](\omega_1 dy \wedge dt + \omega_2 dx \wedge dt + \omega_3 dx \wedge dy) = \varphi_\# [\![Q]\!](\alpha \wedge \xi + d\alpha \wedge \eta)$$

$$= \lim_{\varepsilon \to 0} \int_Q \varphi_\varepsilon^*(\alpha \wedge \xi + \eta \, d\alpha)$$

$$= \lim_{\varepsilon \to 0} \int_Q \varphi_\varepsilon^*(\alpha \wedge \xi) + \lim_{\varepsilon \to 0} \int_Q \eta \circ \varphi_\varepsilon \, d(\varphi_\varepsilon^* \alpha)$$

$$= 0 \qquad \blacksquare$$

$$\int_Q \varphi_\varepsilon^*(\alpha \wedge \xi) \le C \, \|\varphi_\varepsilon^* \alpha\|_\infty \|\varphi_\varepsilon^* \xi\|_\infty$$

$$\le C \, \varepsilon^{2\gamma-1} \varepsilon^{\gamma-1} \to 0$$

$$\int_Q \eta \circ \varphi_\varepsilon \, d(\varphi_\varepsilon^* \alpha) = \int_{\partial Q} \varphi_\varepsilon^*(\eta \, \alpha) - \int_Q \varphi_\varepsilon^*(\alpha \wedge d\eta)$$

$$\left| \int_Q \eta \circ \varphi_\varepsilon \, d(\varphi_\varepsilon^* \alpha) \right| \le C\varepsilon^{2\gamma-1} + C\varepsilon^{3\gamma-2}$$

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