Instantaneous Blowup

Prof. Jerry Goldstein
University of Memphis

The parabolic partial differential equation

$$\frac{\partial u}{\partial t} = \Delta u + \frac{c}{|x|^2} u \tag{1}$$

for $x \in \mathbb{R}^N$, $t > 0$ has remarkable scaling and other properties. It was the first example of instantaneous blowup. For

$$c \leq C(N) = \left(\frac{N-2}{2}\right)^2,$$

(1) has many positive solutions, but for $c > C(N)$, (1) has no nonnegative solutions other than zero. Moreover, suppose $c > C(N)$ and $u(x,0) = f(x) \geq 0$, where f is not 0 a.e. and f is a good initial condition (when $c = 0$). Replace $V(x) = \frac{c}{|x|^2}$ by

$$V_n(x) = \min\left\{V(x), cn^2\right\}.$$

Then $\{V_n\}$ is an increasing sequence of bounded functions, and let $u_n(x,t)$ be the corresponding positive solution of

$$\frac{\partial u}{\partial t} = \Delta u + V_n u.$$

Then

$$u_n(x,t) \longrightarrow +\infty \qquad \text{as } n \longrightarrow \infty$$

for all $x \in \mathbb{R}^N$, $t > 0$. This result is due to P. Baras and J. Goldstein (1984) and was the first example of instantaneous blowup.

The same result holds when $\mathbb{R}^N$ is replaced by the Heisenberg group $\mathbb{H}^N$. The underlying space is topologically $\mathbb{R}^{2N+1}$, but the corresponding Laplacian is the sum of $2N$ squares of vector fields, not $2N+1$ vector fields; thus, the Heisenberg Laplacian is not uniformly elliptic. Nevertheless, with the appropriate potential for the Heisenberg group, instantaneous blowup still holds. The difficult proof is based on extending DeGiorgi-Nash -Moser theory from $\mathbb{R}^N$ to $\mathbb{H}^N$. This result will appear soon in the *Annali della Scuola Normale Superiore di Pisa.* The authors are Gisèle and Jerry Goldstein, Alessia Kogoi, Abdelaziz Rhandi and Cristian Tacellli. The complicated proofs will be discussed. Related results for the modified Ornstein-Uhlenbeck equation and some nonlinear equations will also be presented.