Instantaneous Blowup

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The parabolic partial differential equation

\[ \frac{\partial u}{\partial t} = \Delta u + \frac{c}{|x|^2}u \]  \hspace{1cm} (1)

for \( x \in \mathbb{R}^N, \ t > 0 \) has remarkable scaling and other properties. It was the first example of instantaneous blowup. For

\[ c \leq C(N) = \left( \frac{N-2}{2} \right)^2, \]

(1) has many positive solutions, but for \( c > C(N) \), (1) has no nonnegative solutions other than zero. Moreover, suppose \( c > C(N) \) and \( u(x,0) = f(x) \geq 0 \), where \( f \) is not 0 a.e. and \( f \) is a good initial condition (when \( c = 0 \)). Replace \( V(x) = \frac{c}{|x|^2} \) by

\[ V_n(x) = \min \{ V(x), cn^2 \}. \]

Then \( \{V_n\} \) is an increasing sequence of bounded functions, and let \( u_n(x,t) \) be the corresponding positive solution of

\[ \frac{\partial u}{\partial t} = \Delta u + V_n u. \]

Then

\[ u_n(x,t) \to +\infty \quad \text{as} \quad n \to \infty \]

for all \( x \in \mathbb{R}^N, \ t > 0 \). This result is due to P. Baras and J. Goldstein (1984) and was the first example of instantaneous blowup.

The same result holds when \( \mathbb{R}^N \) is replaced by the Heisenberg group \( \mathbb{H}^N \). The underlying space is topologically \( \mathbb{R}^{2N+1} \), but the corresponding Laplacian is the sum of \( 2N \) squares of vector fields, not \( 2N+1 \) vector fields; thus, the Heisenberg Laplacian is not uniformly elliptic. Nevertheless, with the appropriate potential for the Heisenberg group, instantaneous blowup still holds. The difficult proof is based on extending DeGiorgi-Nash-Moser theory from \( \mathbb{R}^N \) to \( \mathbb{H}^N \). This result will appear soon in the *Annali della Scuola Normale Superiore di Pisa*. The authors are Gisèle and Jerry Goldstein, Alessia Kogoi, Abdelaziz Rhandi and Cristian Tacelli. The complicated proofs will be discussed. Related results for the modified Ornstein-Uhlenbeck equation and some nonlinear equations will also be presented.