Integrating Functions by Matrix Multiplication

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Linear Transformation

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$T : V \to W$ is a *linear transformation* if

1. $T(v_1 + v_2) = Tv_1 + Tv_2$, for all $v_1, v_2 \in V$; and
2. $T(kv) = kTv$, for all $k \in \mathbb{F}$ and for all $v \in V$.

Linear Transformation Example

Suppose that $V = \mathbb{R}^4$ and $W = \mathbb{R}^3$. Let $T : V \to W$ be defined by:

$$T \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} x + 2y \\ w \\ z \end{bmatrix} \quad \text{for all } v = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in V$$

Linear Transformation Example

$$T\left(\begin{bmatrix}x_1\\y_1\\z_1\\w_1\end{bmatrix}+\begin{bmatrix}x_2\\y_2\\z_2\\w_2\end{bmatrix}\right)=T\begin{bmatrix}x_1+x_2\\y_1+y_2\\z_1+z_2\\w_1+w_2\end{bmatrix}=\begin{bmatrix}(x_1+x_2)+2(y_1+y_2)\\w_1+w_2\\z_1+z_2\end{bmatrix}$$

$$=\begin{bmatrix}x_1+2y_1\\w_1\\z_1\end{bmatrix}+\begin{bmatrix}x_2+2y_2\\w_2\\z_2\end{bmatrix}=T\begin{bmatrix}x_1\\y_1\\z_1\\w_1\end{bmatrix}+T\begin{bmatrix}x_2\\y_2\\z_2\\w_2\end{bmatrix}$$

Linear Transformation Example

$$T\left(k\begin{bmatrix}x\\y\\z\\w\end{bmatrix}\right)=T\begin{bmatrix}kx\\ky\\kz\\kw\end{bmatrix}=\begin{bmatrix}kx+2ky\\kw\\kz\end{bmatrix}$$

$$=\begin{bmatrix}k(x+2y)\\kw\\kz\end{bmatrix}=k\begin{bmatrix}x+2y\\w\\z\end{bmatrix}=kT\begin{bmatrix}x\\y\\z\\w\end{bmatrix}$$

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Write

$$\begin{aligned} Tv_1 &= a_{1,1}w_1 + \cdots + a_{m,1}w_m \\ Tv_2 &= a_{1,2}w_1 + \cdots + a_{m,2}w_m \\ &\vdots \\ Tv_n &= a_{1,n}w_1 + \cdots + a_{m,n}w_m \end{aligned}$$

Representing Linear Transformations with Matrices

Let

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}_{m \times n}$$

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Then, for any $v \in V$ with coordinates $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$,

$$[Tv] = A[v]$$

Revisiting Linear Transformation Example

$V = \mathbb{R}^4$, $W = \mathbb{R}^3$, and $T : V \to W$ by $T \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} x + 2y \\ w \\ z \end{bmatrix}$.

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Let $A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

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Then, for any $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in \mathbb{R}^4$, $T \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$.

Integration Example

Find $\int \left(2e^x + 3xe^x - 4x^2e^x\right)\, dx.$

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$$\begin{aligned} &\int \left(2e^x + 3xe^x - 4x^2e^x\right)\, dx \\ &= 2\left(\int e^x\, dx\right) + 3\left(\int xe^x\, dx\right) - 4\left(\int x^2e^x\, dx\right) \end{aligned}$$

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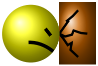
$=$ 

Integration Example - New Approach

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Let $D : V \to V$ be the differentiation transformation. (Note that it is important that $D(V) \subset V$).

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Let $V = \text{span}\{e^x, xe^x, x^2e^x\}$.

Let $D : V \to V$ be the differentiation transformation. (Note that it is important that $D(V) \subset V$).

The matrix that represents this transformation is

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

The Inverse of Differentiation

Note that if $D : V \to V$ is the differentiation transformation, then $D^{-1} : V \to V$ is the integration transformation (where $+C = 0$, so that the transformation is linear).

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FACT

If A represents the linear transformation D, then the inverse matrix A^{-1} represents the inverse transformation D^{-1}.

In our case, if $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$, then $A^{-1} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$.

Finishing the Previous Example

Previously, we wanted to find $\int \left(2e^x + 3xe^x - 4x^2e^x\right)\ dx$.

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The coordinates of this vector under the given basis are $\begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}$.

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The coordinates of this vector under the given basis are $\begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}$.

Therefore, to find our integral, we need to find $D^{-1}(2e^x + 3xe^x - 4x^2e^x)$.

The Calculation and the Interpretation

$$D^{-1}(2e^x + 3xe^x - 4x^2e^x)$$
$$= \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \\ -4 \end{bmatrix}$$

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Therefore, we have that

$$\int \left(2e^x + 3xe^x - 4x^2e^x\right) \, dx = -9e^x + 11xe^x - 4x^2e^x$$

The Extension

This technique now works quickly for any integral of this form.

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No need to restart!

Answer

$$D^{-1}(11xe^x + \frac{2}{3}x^2e^x) = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 11 \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} -29/3 \\ 29/3 \\ 2/3 \end{bmatrix}.$$

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Answer

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Therefore, $\int \left(11xe^x + \frac{2}{3}x^2e^x\right)\, dx = -\frac{29}{3}e^x + \frac{29}{3}xe^x + \frac{2}{3}x^2e^x$.

The Problem

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The problem is that $D(e^{x^2}) = 2xe^{x^2}$, $D(xe^{x^2}) = (1 + 2x^2)e^{x^2}$, $D(x^2e^{x^2}) = (2x + 2x^3)e^{x^2}$, $\ldots$, etc.

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Matrices (where B is A^{-1} from before):

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(Notice the pattern to each matrix - **THIS IS THE WINNING STRATEGY!**)

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Therefore,

$$\int (2e^x \sin x - 4e^x \cos x)\ dx = -e^x \sin x - 3e^x \cos x.$$

Thank you, and . . .
Ask me more about Math 0280!