The Frobenius Coin Problem A Geometric Proof for Two Variables

Derek Orr - University of Pittsburgh



Undergraduate Math Seminar - University of Pittsburgh

October 24, 2017



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The Frobenius Coin Problem

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An Example Statement Preliminaries

Consider the scoring in the NFL, before 1994:



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An Example Statement Preliminaries

Consider the scoring in the NFL, before 1994:

► Teams can either get 7 points or 3 points.



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An Example Statement Preliminaries

Consider the scoring in the NFL, before 1994:

- ► Teams can either get 7 points or 3 points.
- What is the maximum score that cannot be attained?



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An Example Statement Preliminaries

Consider the scoring in the NFL, before 1994:

- ► Teams can either get 7 points or 3 points.
- ► What is the maximum score that cannot be attained? Answer: 11.



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An Example Statement Preliminaries

Consider the scoring in the NFL, before 1994:

- ► Teams can either get 7 points or 3 points.
- ► What is the maximum score that cannot be attained? Answer: 11. Why?



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An Example Statement Preliminaries

We are looking at 7x + 3y = n.



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The Problem An Example Proof Statement Extensions Preliminaries

We are looking at 7x + 3y = n. We can see that you can't achieve n = 11 using only 7 and 3 (and positive values of x and y).



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An Example Statement Preliminaries

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The ProblemAn ExampleProofStatementExtensionsPreliminaries

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We are looking at 7x + 3y = n. We can see that you can't achieve n = 11 using only 7 and 3 (and positive values of x and y). Let's check if it's the smallest: 12 = 0 * 7 + 4 * 313 = 1 * 7 + 2 * 3



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We are looking at 7x + 3y = n. We can see that you can't achieve n = 11 using only 7 and 3 (and positive values of x and y).
Let's check if it's the smallest:
12 = 0 * 7 + 4 * 3
13 = 1 * 7 + 2 * 3
14 = 2 * 7 + 0 * 3
```



The Frobenius Coin Problem

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The Frobenius Coin Problem



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Let's check if it's the smallest:
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So, it probably works for all numbers greater than 11...



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We are looking at 7x + 3y = n. We can see that you can't achieve n = 11 using only 7 and 3 (and positive values of x and y). Let's check if it's the smallest: 12 = 0 * 7 + 4 * 313 = 1 * 7 + 2 * 314 = 2 * 7 + 0 * 315 = 0 * 7 + 5 * 3

So, it probably works for all numbers greater than 11... (not a proof!)



An Example Statement Preliminaries

Frobenius Coin Problem: What is the largest number *n* such that ax + by = n has no solutions for $x \ge 0$ and $y \ge 0$?



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An Example Statement Preliminaries

Frobenius Coin Problem: What is the largest number *n* such that ax + by = n has no solutions for $x \ge 0$ and $y \ge 0$?

Since *n* depends on *a* and *b*, call it a function g(a, b).



The Frobenius Coin Problem



Frobenius Coin Problem: What is the largest number *n* such that ax + by = n has no solutions for $x \ge 0$ and $y \ge 0$?

- ▶ Since *n* depends on *a* and *b*, call it a function g(a, b).
- ► g(a, b) is called the Frobenius Number.



The Frobenius Coin Problem

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- g(3,7) = g(7,3) = 11.



Frobenius Coin Problem: What is the largest number *n* such that ax + by = n has no solutions for $x \ge 0$ and $y \ge 0$?

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- ► g(a, b) is called the Frobenius Number.
- g(3,7) = g(7,3) = 11.

Claim: g(a, b) = ab - a - b.



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Some notation:



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Some notation:

pcd(a, b) is the greatest common divisor of a and b. (e.g., gcd(6, 33) = 3)



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An Example Statement Preliminaries

Some notation:

- ▶ gcd(a, b) is the greatest common divisor of a and b. (e.g., gcd(6, 33) = 3)
- ▶ $\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, ...\}$ is the set of all integers.



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Some notation:

- ▶ gcd(a, b) is the greatest common divisor of a and b. (e.g., gcd(6, 33) = 3)
- ▶ $\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, ...\}$ is the set of all integers.
- $\mathbb{N} = \{1, 2, 3, ...\}$ is the set of all positive integers.



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The equation ax + by = n is called a Linear Diophantine Equation.



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The equation ax + by = n is called a Linear Diophantine Equation. Our goal is to find nonnegative integers x and y that satisfy this for given a, b, and n.



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Theorem: If (x_0, y_0) is a solution of ax + by = n, then all solutions are given by $\{(x, y) : x = x_0 + bt, y = y_0 - at, t \in \mathbb{Z}\}$



The Frobenius Coin Problem



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"Proof" (plug it in):





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"Proof" (plug it in): $a(x_0 + bt) + b(y_0 - at)$





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The Frobenius Coin Problem



Theorem: If (x_0, y_0) is a solution of ax + by = n, then all solutions are given by $\{(x, y) : x = x_0 + bt, y = y_0 - at, t \in \mathbb{Z}\}$

"Proof" (plug it in): $a(x_0 + bt) + b(y_0 - at) = ax_0 + abt + by_0 - bat = ax_0 + by_0 = n$ by definition of (x_0, y_0) .



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Proof of FCP.



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Proof of FCP. Consider the equation ax + by = n for $n \in \mathbb{N}$ and a, b > 1.



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Proof of FCP.

Consider the equation ax + by = n for $n \in \mathbb{N}$ and a, b > 1.

► Let gcd(a, b) = 1 (Otherwise, there is always some n that will yield no solutions)





Proof of FCP.

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► Let gcd(a, b) = 1 (Otherwise, there is always some n that will yield no solutions)

• Solving for y above implies
$$y = \frac{n - ax}{b} = \frac{n}{b} - \frac{a}{b}x$$
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Proof of FCP.

Consider the equation ax + by = n for $n \in \mathbb{N}$ and a, b > 1.

- ► Let gcd(a, b) = 1 (Otherwise, there is always some n that will yield no solutions)
- ► Solving for y above implies $y = \frac{n ax}{b} = \frac{n}{b} \frac{a}{b}x$.
- ► Focus on the first quadrant (i.e. x, y ≥ 0) and look what happens when n increases.



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The Problem Proof Extensions Upper Bound Lower Bound In Between

Recall if (x_0, y_0) is a solution, then $(x_0 + bt, y_0 - at)$ is a solution for $t \in \mathbb{Z}$.



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Recall if (x_0, y_0) is a solution, then $(x_0 + bt, y_0 - at)$ is a solution for $t \in \mathbb{Z}$. Consecutive solutions are given by (x_0, y_0) and $(x_0 + b, y_0 - a)$.



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$$L = \sqrt{([x_0 + b] - x_0)^2 + ([y_0 - a] - y_0)^2}$$



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$$L = \sqrt{([x_0 + b] - x_0)^2 + ([y_0 - a] - y_0)^2} = \sqrt{a^2 + b^2}$$



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Let L be the distance between two consecutive solutions.

$$L = \sqrt{([x_0 + b] - x_0)^2 + ([y_0 - a] - y_0)^2} = \sqrt{a^2 + b^2}$$

If you start at some point on the line ax + by = n and travel a distance $\sqrt{a^2 + b^2}$, you *must* pass through an integral solution.



Upper Bound Lower Bound In Between

(Upper bound)



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Upper Bound Lower Bound In Between

(Upper bound)

Let $S = \{(x, y) : ax + by = n; x, y \ge 0\}$ be the section of the line that lies in the first quadrant.



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(Upper bound)

Let $S = \{(x, y) : ax + by = n; x, y \ge 0\}$ be the section of the line that lies in the first quadrant. Let's compute the length of S, denoted by |S|.



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The Problem Proof Extensions Upper Bound Lower Bound In Between

Along the line ax + by = n, S has two endpoints, the x and y intercepts.



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The Problem Upper Bound Proof Lower Bound Extensions In Between

Along the line ax + by = n, S has two endpoints, the x and y intercepts. These are given by $\left(0, \frac{n}{b}\right)$ and $\left(\frac{n}{a}, 0\right)$.



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Along the line ax + by = n, S has two endpoints, the x and y intercepts. These are given by $\left(0, \frac{n}{b}\right)$ and $\left(\frac{n}{a}, 0\right)$. Using the distance formula, $|S| = \sqrt{\left(\frac{n}{a} - 0\right)^2 + \left(0 - \frac{n}{b}\right)^2}$



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Along the line ax + by = n, S has two endpoints, the x and y intercepts. These are given by $\left(0, \frac{n}{b}\right)$ and $\left(\frac{n}{a}, 0\right)$. Using the distance formula, $|S| = \sqrt{\left(\frac{n}{a} - 0\right)^2 + \left(0 - \frac{n}{b}\right)^2}$

$$= n\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$$



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Along the line ax + by = n, S has two endpoints, the x and y intercepts. These are given by $\left(0, \frac{n}{b}\right)$ and $\left(\frac{n}{a}, 0\right)$. Using the distance formula, $|S| = \sqrt{\left(\frac{n}{a} - 0\right)^2 + \left(0 - \frac{n}{b}\right)^2}$ $= n\sqrt{\frac{1}{a^2} + \frac{1}{b^2}} = \frac{n}{ab}\sqrt{b^2 + a^2} = \frac{n}{ab}L.$



The Frobenius Coin Problem

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Along the line ax + by = n, S has two endpoints, the x and y intercepts. These are given by $\left(0, \frac{n}{b}\right)$ and $\left(\frac{n}{a}, 0\right)$. Using the distance formula, $|S| = \sqrt{\left(\frac{n}{a} - 0\right)^2 + \left(0 - \frac{n}{b}\right)^2}$ $= n\sqrt{\frac{1}{a^2} + \frac{1}{b^2}} = \frac{n}{ab}\sqrt{b^2 + a^2} = \frac{n}{ab}L.$

Hence, for any $n \ge ab$, $|S| \ge L = \sqrt{a^2 + b^2}$, meaning there must be an integral solution on S.



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Upper Bound Lower Bound In Between



Figure: Example with a = 5, b = 7, and n = 35.

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Upper Bound Lower Bound In Between



Figure: Example with a = 5, b = 7, n = 35, and n = 43.



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Figure: Example with a = 5, b = 7, n = 35, n = 43, and n = 54.

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Upper Bound Lower Bound In Between



Figure: Example with a = 5, b = 7, n = 35, n = 43, and n = 54.

Thus, g(a, b) < ab.

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Upper Bound Lower Bound In Between

(Lower Bound)



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Upper Bound Lower Bound In Between

(Lower Bound) Consider n = ab - a - b.



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Upper Bound Lower Bound In Between

(Lower Bound) Consider n = ab - a - b. So,

ax + by = ab - a - b



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The Problem Proof Extensions Upper Bound Lower Bound In Between

(Lower Bound) Consider n = ab - a - b. So,

ax + by = ab - a - b = a(b - 1) + b(-1)



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The Problem Proof Extensions Upper Bound Lower Bound In Between

(Lower Bound) Consider n = ab - a - b. So,

 $\overline{ax + by} = ab - a - b = a(b - 1) + b(-1) = a(-1) + b(a - 1).$



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The Problem Upper Bound Proof Lower Bound Extensions In Between

(Lower Bound) Consider n = ab - a - b. So,

ax + by = ab - a - b = a(b - 1) + b(-1) = a(-1) + b(a - 1).

(-1, a − 1) and (b − 1, −1) are integer solutions to this equation.



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The Problem Upper Bound Proof Lower Bound Extensions In Between

(Lower Bound) Consider n = ab - a - b. So,

ax + by = ab - a - b = a(b - 1) + b(-1) = a(-1) + b(a - 1).

- ► (-1, a 1) and (b 1, -1) are integer solutions to this equation.
- ► Not only that, but they are consecutive! (Remember $(x_0, y_0) \implies (x_0 + b, y_0 a)$)



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Figure: Example with a = 5, b = 7.



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Upper Bound Lower Bound In Between



Figure: Example with a = 5, b = 7.

Since a solution must lie in the first quadrant, $g(a, b) \ge ab - a - b$.

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Now we have $ab - a - b \le g(\overline{a, b}) < ab$.



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Now we have $ab - a - b \le g(a, b) < ab$. What if ab - a - b < n < ab?



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Figure: Example with a = 5, b = 7.



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Figure: Example with a = 5, b = 7.

Both of these lines are parallel (the slope is $-\frac{a}{b}$).



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Figure: Example with a = 5, b = 7.

Both of these lines are parallel (the slope is $-\frac{a}{b}$). So the shape connecting all four points is a parallelogram.

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Figure: Example with a = 5, b = 7.



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Figure: Example with a = 5, b = 7.

Since the longer side is $\sqrt{a^2 + b^2}$, for ab - a - b < n < ab, there must be a solution inside the parallelogram for each *n*.

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Figure: Example with a = 5, b = 7. (lower triangle)



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Figure: Example with a = 5, b = 7. (lower triangle)

Lower Triangle: Any solution inside will have -1 < y < 0.

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Figure: Example with a = 5, b = 7. (upper triangle)



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Figure: Example with a = 5, b = 7. (upper triangle)

Upper Triangle: Any solution inside will have -1 < x < 0.

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Figure: Example with a = 5, b = 7.

Thus, the integer solutions must all be in the first quadrant.



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Figure: Example with a = 5, b = 7.

Thus, the integer solutions must all be in the first quadrant. Hence, g(a, b) = ab - a - b.

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Figure: Example with a = 5, b = 7.

Thus, the integer solutions must all be in the first quadrant. Hence, g(a, b) = ab - a - b.

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3 Variables Variations

What about 3 variables: ax + by + cz = d?



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What about 3 variables: ax + by + cz = d?

► This is the equation of a plane, instead of a line.





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What about 3 variables: ax + by + cz = d?

- This is the equation of a plane, instead of a line.
- Harder than one may think.



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3 Variables Variations

McNugget Numbers (Henri Picciotto, 1980s)



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3 Variables Variations

McNugget Numbers (Henri Picciotto, 1980s)

► Original boxes had 6, 9, and 20 nuggets.



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McNugget Numbers (Henri Picciotto, 1980s)

- ► Original boxes had 6, 9, and 20 nuggets.
- Worked out the largest non-McNugget number on a napkin.



McNugget Numbers (Henri Picciotto, 1980s)

- ► Original boxes had 6, 9, and 20 nuggets.
- Worked out the largest non-McNugget number on a napkin.
- g(6,9,20) = 43.



3 Variables Variations

Formula for g(a, b, c)?



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3 Variables Variations

Formula for g(a, b, c)?

► Must be commutative.



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3 Variables Variations

Formula for g(a, b, c)?

- Must be commutative.
- ► $g(a, b, c) \leq g(a, b)$.



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3 Variables Variations

There is **no** known explicit formula for g(a, b, c).



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3 Variables Variations

There is **no** known explicit formula for g(a, b, c). (Though there are upper and lower bounds...)



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3 Variables Variations

Results:



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3 Variables Variations

Results:

The largest integer that cannot be represented as a nonnegative linear combination of *a* and *b* is g(a,b) = ab - a - b = (a-1)(b-1) - 1.



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3 Variables Variations

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For 3 or more variables:





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3 Variable Variations

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For 3 or more variables:

Lower bound (J. L. Davison, 1994): g(a, b, c) ≥ √3abc − a − b − c.





3 Variables Variations

Results:

The largest integer that cannot be represented as a nonnegative linear combination of *a* and *b* is g(a,b) = ab - a - b = (a-1)(b-1) - 1.

For 3 or more variables:

- ► Lower bound (J. L. Davison, 1994): g(a, b, c) ≥ √3abc - a - b - c.
- ▶ Upper bound (Erdős, Graham in Acta Arithmetica 1972): $g(a_1,...,a_n) \le 2a_{n-1} \lfloor \frac{a_n}{n} \rfloor - a_n$ where $a_1 < a_2 < ... < a_n$.





Extensions: Arithmetic sequences: Given *a*, *d*, *s* with gcd(a, d) = 1,

$$g(a, a+d, a+2d, \ldots, a+sd) = \left(\left\lfloor \frac{a-2}{s} \right\rfloor + 1\right)a + ad - a - d.$$



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Extensions: Arithmetic sequences: Given *a*, *d*, *s* with gcd(a, d) = 1,

$$g(a, a+d, a+2d, \ldots, a+sd) = \left(\left\lfloor \frac{a-2}{s} \right\rfloor + 1\right)a + ad - a - d.$$

Geometric sequences: Given m, n, p, with gcd(m, n) = 1,

$$g(m^{p}, m^{p-1}n, m^{p-2}n^{2}, \dots, mn^{p-1}, n^{p})$$

= $n^{p-1}(mn - m - n) + \frac{m^{2}(n-1)(m^{p-1} - n^{p-1})}{m - n}$

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3 Variables Variations

Extensions:

There are (a-1)(b-1)/2 = (g(a,b)+1)/2 natural numbers that will never be attained. For 3 and 7, $\{1,2,4,5,8,11\}$ are never achieved, a total of (11+1)/2 = 6 numbers.



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3 Variables Variations

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What is the largest number *n* that has a *unique* solution (x, y) to ax + by = n? (Call it h(a, b))



3 Variables Variations

Extensions:

There are (a-1)(b-1)/2 = (g(a,b)+1)/2 natural numbers that will never be attained. For 3 and 7, $\{1, 2, 4, 5, 8, 11\}$ are never achieved, a total of (11+1)/2 = 6 numbers.

What is the largest number *n* that has a *unique* solution (x, y) to ax + by = n? (Call it h(a, b)) Did some coding... h(a, b) = 2ab - a - b. For example, with 3 and 7, 32 is the largest number with a unique pairing:

32 = 3 * 6 + 7 * 2. All larger numbers have more than one solution. 33 = 3 * 11 + 7 * 0 = 3 * 4 + 7 * 3. This can be good for making bets...

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3 Variables Variations

Thank You!



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