"THE MAZUR PRODUCT OF CONVERGENT SEQUENCES IS NOT ONTO. 3."

FUNCTIONAL ANALYSIS SEMINAR

TUESDAY 25/FEBRUARY/2020, 3:00 PM - 3:50 PM
ROOM 321 THACKERAY

CHRIS LENNARD

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Abstract. We present a discussion of a 2009 solution of Mazur's Problem 8 from the Scottish Book by [L/Dan Radelet, JMAA 350, pp. 384-392]. It is well known that the Cesàro mapping C on *the space of convergent sequences of complex numbers* c ,

$$C : x = (x_n)_{n \geq 0} \mapsto \left(\frac{x_0 + x_1 + \cdots + x_n}{n+1} \right)_{n \geq 0} \quad ,$$

preserves the limit $L = \lambda(x) := \lim_{n \longrightarrow \infty} \ x_n$, but is **not** onto.

Put differently, C maps c into c, $\lambda(Cx) = \lambda(x)$, and there exists $z = (z_n)_{n \geq 0}$ in c such that $z \neq Cx$, for all $x \in c$. Indeed,

$$z = (\,1,\, 0\,,\, 1/3\,,\, 0\,,\, 1/5\,,\, 0\,,\, 1/7\,,\, 0\,, \ldots\,)$$

is such a sequence.

In the Scottish Book, Stanisław Mazur considered an extension of the Cesàro mapping to a bilinear product $\boxtimes$ on c : for all $x, y \in c$,

$$x \boxtimes y := \left(\frac{x_0\, y_n + x_1\, y_{n-1} + \cdots + x_n\, y_0}{n+1} \right)_{n \geq 0} \quad .$$

It turns out that for all $x, y \in c$, $x \boxtimes y \in c$ and

$$\lambda(x \boxtimes y) = \lambda(x)\, \lambda(y) \ .$$

Note also that $x \boxtimes \mathbb{I} = Cx$, where $\mathbb{I} := (1, 1, 1, \ldots, 1, \ldots)$. In the Scottish Book, Problem 8, Professor Mazur asked *whether the mapping* $\boxtimes : c \times c \longrightarrow c$ *is onto?* The prize for solution, written in the margin, was *5 small beers.*

This question was solved in the *negative* in the 1980's:
[*There exists* $z \in c$ *such that* $z \neq x \boxtimes y$, *for all* $x, y \in c$]. It was solved independently by P.P.B. Eggermont and Y.J. Leung, as well as S. Kwapień and A. Pełczyński, and also by V.V. Peller.

Later, L/Dan Radelet (2009) came up with another solution to Mazur's problem 8, by solving a more general question. Note that to solve Mazur's problem in the negative, it is sufficient to showing that $\boxtimes : c \times c_0 \longrightarrow c_0$ is not onto. Here, c_0 is the space of all real sequences that converge to zero. L/Radelet showed that [*There exists* $z \in c_0$ *such that* $z \neq x \boxtimes y$, *for all* $x \in \mathcal{H}^2$ *and for all* $y \in \mathcal{H}_0^2$], where $\mathcal{H}^2$ is a larger space than c and $\mathcal{H}_0^2$ is a larger space than c_0.

We will discuss all this, and the ideas behind our solution (including plenty of illustrative examples), over the span of a few Functional Analysis Seminars.