Talk Undergraduate Seminar:

Euler's zeta and double zeta values and rational zeta series representations for Apéry's constant $\zeta(3)$.

The Riemann zeta function was defined by Riemann himself back in 1859 by

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}, \ \mathrm{Re}\, s>1.$$

Although this looks quite easy, there are still a lot to understand about this function!
Everything started from a letter of Bernoulli to Euler about the evaluation of

$$\zeta(2)=\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{1}{1^2}+\frac{1}{2^2}+\dots+\frac{1}{n^2}+\dots$$

which surprisingly in a way led to $\frac{\pi^2}{6}$. This major breakthrough was done by Euler in 1734. We shall prove this result, but we give a different proof than Euler's initial proof.
Our proof will have a bit differential equations flavour. Let's start with the function $y:(-1,1)\to\mathbb{R}$, $y(x)=\arcsin^2 x$. If we differentiate it once, we get $y'(x)=2\arcsin x\cdot\frac{1}{\sqrt{1-x^2}}$ or equivalently we can write

$\sqrt{1-x^2}\, y'(x) = 2\arcsin x$. Differentiating one more time, we have

$(\sqrt{1-x^2}\, y'(x))' = (2\arcsin x)'$ which gives us

$\underbrace{(\sqrt{1-x^2})'}_{=} y'(x) + \sqrt{1-x^2}\, y''(x) = 2\frac{1}{\sqrt{1-x^2}}$ or equivalently

$\frac{1}{2\sqrt{1-x^2}}\cdot(-2x)\cdot y'(x) + \sqrt{1-x^2}\, y''(x) = \frac{2}{\sqrt{1-x^2}}$ or even better

by multiplying with $\sqrt{1-x^2}$, we obtain the following

$$\begin{cases} (1-x^2)\, y''(x) - x y'(x) - 2 = 0 \\ y(0) = y'(0) = 0 \end{cases}$$

$\rightsquigarrow$ initial value problem!

One can look for a power series solution,

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Differentiating once and even twice, we get

$$y'(x) = \sum_{n=0}^{\infty} a_n \cdot n x^{n-1} = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$y''(x) = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}.$$

Rearranging in our IVP, we derive

$$2a_2 + 3a_3 \cdot x + \sum_{n=2}^{\infty} \left((n+2)(n+1)a_{n+2} - n^2 a_n\right)x^n = 2.$$

Clearly, $a_2 = 1$, $a_3 = 0$, and for $n \geq 2$, $a_{2n+1} = 0$, and

$$a_{n+2} = \frac{n^2}{(n+2)(n+1)} \cdot a_n, \quad n \geq 2.$$

This easily implies that

$$a_{2n} = \frac{\left(2^{n-1}(n-1)!\right)^2}{(2n)(2n-1)\cdot \ldots \cdot 4 \cdot 3} = \frac{1}{2} \cdot \frac{4^n}{n^2 \binom{2n}{n}}.$$

Therefore, we have

$$\boxed{y(x) = \arcsin^2 x = \sum_{n=1}^{\infty} \frac{2^{2n-1}}{n^2 \binom{2n}{n}} \cdot x^{2n}, \quad |x| \leq 1.}$$

Taking $x = \sin t$, we get

$$t^2 = \sum_{n=1}^{\infty} \frac{2^{2n-1}}{n^2 \binom{2n}{n}} \cdot \sin^{2n} t.$$

Integrating from 0 to $\frac{\pi}{2}$, we obtain

$$\int_0^{\frac{\pi}{2}} t^2 \, dt = \sum_{n=1}^{\infty} \frac{2^{2n-1}}{n^2 \binom{2n}{n}} \cdot \int_0^{\frac{\pi}{2}} \sin^{2n} t \, dt.$$

By Wallis' integral formula,

$$\int_0^{\frac{\pi}{2}} \sin^{2n} t \, dt = \frac{\pi \binom{2n}{n}}{2^{2n+1}},$$ we finally have

$$\frac{\pi^3}{24} = \sum_{n=1}^{\infty} \frac{\cancel{2^{2n-1}}}{n^2 \cancel{\binom{2n}{n}}} \cdot \frac{\pi \cancel{\binom{2n}{n}}}{\cancel{2^{2n+1}}_{4}} \text{ or}$$

$$\boxed{\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}} \quad !$$

Remark. Euler also computed $\zeta(4) = \frac{\pi^4}{90}$, $\zeta(6) = \frac{\pi^6}{945}$, - etc!

Now, let us define the following double zeta values:

$$\zeta(a,b) := \sum_{m>n\geq 1} \frac{1}{m^a n^b}, \quad a,b \in \mathbb{Z}, \ a \geq 2, \ b \geq 1$$

In 1775, Euler proved even more:

$$\boxed{\zeta(2,1) = \zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}}$$

$$\zeta(2,1) = \sum_{n=1}^{\infty} \frac{1}{n^2} \cdot \sum_{m=1}^{n-1} \frac{1}{m}$$

Let's give one of Euler's proofs! First, let's start with the following

Lemma.

$$I = \int_0^1 u^{k-1}(-\log u)\,du = \frac{1}{k^2}, \ k \geq 1.$$

Proof of the lemma.

$$I = -\frac{1}{K}\cdot\int_0^1 (u^K)' \log u \, du = -\frac{1}{K}\left(u^K\cdot \log u\Big|_0^1 - \int_0^1 u^K \cdot \frac{1}{u} du\right.$$

$$= \frac{1}{K}\cdot\int_0^1 u^{K-1} du = \frac{1}{K^2}. \quad \blacksquare$$

Now, we have:

$$\zeta(2,1) = \sum_{k>n\geq 1} \frac{1}{k^2 n} = \sum_{n=1}^{\infty} \frac{1}{n}\cdot\sum_{k>n}\frac{1}{K^2} = \sum_{n=1}^{\infty}\frac{1}{n}\cdot \sum_{k>n}\int_0^1 u^{K-1}(-\log u) du$$

$$= \sum_{n=1}^{\infty}\frac{1}{n}\cdot\int_0^1 (-\log u)\cdot\underbrace{\sum_{k>n} u^{K-1}}_{=\frac{u^n}{1-u}} du =$$

$$= -\int_0^1 \frac{\log u}{1-u}\cdot\underbrace{\sum_{n=1}^{\infty}\frac{u^n}{n}}_{=\log\left(\frac{1}{1-u}\right)} du = \int_0^1 \frac{(-\log u)}{1-u}\cdot \log\left(\frac{1}{1-u}\right) du$$

Using the change of variables $\boxed{1-u=t}$, we have

$$\sum_{k>n\geq 1} \frac{1}{k^2 n} = \int_0^1 \log\frac{1}{1-t}\,(-\log t)\,\frac{dt}{t}$$

$$= \int_0^1 (-\log t) \sum_{n\geq 1} \frac{t^{n-1}}{n}\, dt$$

$$= \sum_{n\geq 1} \frac{1}{n} \int_0^1 (-\log t)\, t^{n-1}\, dt$$

$$= \frac{1}{n^2}$$

$$= \sum_{n\geq 1} \frac{1}{n^3},$$

which is exactly what we wanted prove. □

Remark.

Another important thing about double zeta values is the so-called the stuffle identity:

$$\zeta(a)\cdot\zeta(b) = \zeta(a,b) + \zeta(b,a) + \zeta(a+b).$$

indeed, this can be seen easily in the following way:

$$\zeta(a)\cdot\zeta(b) = \sum_{m\geq 1}\frac{1}{m^a}\cdot\sum_{n\geq 1}\frac{1}{n^b} = \sum_{m,n\geq 1}\frac{1}{m^a n^b} =$$

$$= \Big(\sum_{m>n\geq 1} + \sum_{n>m\geq 1} + \sum_{m=n\geq 1}\Big)\frac{1}{m^a n^b} = \zeta(a,b) + \zeta(b,a) + \zeta(a+b).$$

It is interesting to see that for $a=b=2$, we obtain: $\zeta^2(2) = 2\zeta(2,2) + \zeta(4)$ which will give us $\zeta(2,2) = \frac{\pi^4}{120}$

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EULER AND HIS WORK ON INFINITE SERIES

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For the 300th anniversary of Leonhard Euler's birth

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1. INTRODUCTION

Leonhard Euler is one of the greatest and most astounding icons in the history of science. His work, dating back to the early eighteenth century, is still with us, very much alive and generating intense interest. Like Shakespeare and Mozart, he has remained fresh and captivating because of his personality as well as his ideas and achievements in mathematics. The reasons for this phenomenon lie in his universality, his uniqueness, and the immense output he left behind in papers, correspondence, diaries, and other memorabilia. *Opera Omnia* [E], his collected works and correspondence, is still in the process of completion, close to eighty volumes and 31,000+ pages and counting. A volume of brief summaries of his letters runs to several hundred pages. It is hard to comprehend the prodigious energy and creativity of this man who fueled such a monumental output. Even more remarkable, and in stark contrast to men like Newton and Gauss, is the sunny and equable temperament that informed all of his work, his correspondence, and his interactions with other people, both common and scientific. It was often said of him that he did mathematics as other people breathed, effortlessly and continuously. It was also said (by Laplace) that all mathematicians were his students.

It is appropriate in this, the tercentennial year of his birth, to revisit him and survey his work, its offshoots, and the remarkable vitality of his themes which are still flourishing, and to immerse ourselves once again in the universe of ideas that he has created. This is not a task for a single individual, and appropriately enough, a number of mathematicians are attempting to do this and present a picture of his work and its modern resonances to the general mathematical community. To be honest, such a project is Himalayan in its scope, and it is impossible to do full justice to it. In the following pages I shall try to make a very small contribution to this project, discussing in a sketchy manner Euler's work on infinite series and its modern outgrowths. My aim is to acquaint the generic mathematician with

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some Eulerian themes and point out that some of them are still awaiting complete understanding. Above all, it is the freedom and imagination with which Euler operates that are most compelling, and I would hope that the remarks below have captured at least some of it. For a tribute to this facet of Euler's work, see [C].

The literature on Euler, both personal and mathematical, is huge. The references given at the end are just a fraction of what is relevant and are in no way intended to be complete. However, many of the points examined in this article are treated at much greater length in my book [V], which contains more detailed references. After the book came out, Professor Pierre Deligne, of the Institute for Advanced Study, Princeton, wrote to me some letters in which he discussed his views on some of the themes treated in my book. I have taken the liberty of including here some of his comments that have enriched my understanding of Euler's work, especially on infinite series. I wish to thank Professor Deligne for his generosity in sharing his ideas with me and for giving me permission to discuss them here. I also wish to thank Professor Trond Digernes of the University of Trondheim, Norway, for helping me with electronic computations concerning some continued fractions that come up in Euler's work on summing the factorial-like series.

2. Zeta values

Euler must be regarded as the first master of the theory of infinite series. He created it and was by far its greatest master. Perhaps only Jacobi and Ramanujan may be regarded as being even close. Before Euler entered the mathematical scene, infinite series had been considered by many mathematicians, going back to very early times. However there was no *systematic theory*; people had only very informal ideas about convergence and divergence. Also most of the series considered had only positive terms. Archimedes used the geometric series

$$\frac{4}{3} = 1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \dots$$

in computing, by what he called the method of exhaustion, the area cut off by a secant from a parabola. Leibniz, Gregory, and Newton had also considered various special series, among which the Leibniz evaluation,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots,$$

was a most striking one. In the fourteenth century people discussed the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \dots,$$

and Pietro Mengoli (1625–1686) seems to have posed the problem of finding the sum of the series

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots.$$

This problem generated intense interest, and the Bernoulli brothers, Johann and Jakob, especially the former, appear to have made efforts to find the sum. It came to be known as the *Basel problem*. But all efforts to solve it had proven useless, and even an accurate numerical evaluation was extremely difficult because of the slow decay of the terms. Indeed, since

$$\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} < \frac{1}{n^2} < \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}$$

we have

$$\frac{1}{N+1} < \sum_{n=N+1}^{\infty} \frac{1}{n^2} < \frac{1}{N},$$

so that to compute directly the sum with an accuracy of six decimal places would require taking into account at least a million terms.

Euler's first attack on the Basel problem already revealed how far ahead of everyone else he was. Since the terms of the series decreased very slowly, Euler realized that he had to transform the series into a rapidly convergent one to facilitate easy numerical computation. He did exactly that. To describe his result, let me use modern notation (for brevity) and write

$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots .$$

Then Euler's remarkable formula is

(1) $$\zeta(2) = (\log 2)^2 + 2\sum_{n=1}^{\infty} \frac{1}{n^2.2^n}$$

with

$$\log 2 = \frac{1}{2} + \frac{1}{8} + \frac{1}{24} + \dots = \sum_{n=1}^{\infty} \frac{1}{n.2^n}.$$

The terms in the series are *geometric*, and the one for log 2 is obtained by taking the value $x = \frac{1}{2}$ in the power series for $-\log(1-x)$. However formula (1) lies deeper. Using this he calculated $\zeta(2)$ accurately to six places and obtained the value

$$\zeta(2) = 1.644944\dots .$$

To derive (1) Euler introduced the power series

$$x + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \dots = \sum_{n=1}^{\infty} \frac{x^n}{n^2},$$

which is the *generating function* of the sequence $(1/n^2)$. This is an idea of great significance for him because, throughout his life, especially when he was attempting to build a theory of divergent series, he regarded infinite series as arising out of generating functions by evaluation at special values. In this case the function in question has an *integral representation*: namely

(2) $$\sum_{n=1}^{\infty} \frac{x^n}{n^2} = \mathrm{Li}_2(x)$$

where

$$\mathrm{Li}_2(x) := \int_0^x \frac{-\log(1-t)}{t} dt = \int\int_{0<t_2<t_1<x} \frac{dt_1 dt_2}{t_1(1-t_2)}.$$

It is the first appearance of the *dilogarithm*, a special case of the *polylogarithms* which have been studied recently in connection with *multizeta values* (more about these later). Clearly

$$\zeta(2) = \mathrm{Li}_2(1).$$

The integral representation allowed Euler to transform the series as we shall see now. He obtained the *functional equation*

(3) $$\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) = -\log x \log(1-x) + \mathrm{Li}_2(1),$$

which leads, on taking $x = \frac{1}{2}$, to

$$\zeta(2) = (\log 2)^2 + 2\sum_{n=1}^{\infty} \frac{1}{n^2.2^n}.$$

The formula (3) is easy to prove. We write

$$\zeta(2) = \int_0^u \frac{-\log(1-x)}{x}dx + \int_u^1 \frac{-\log(1-x)}{x}dx.$$

We then change x to $1-x$ in the second integral and integrate it by parts to get (1). More than the specific result, the significance of Euler's result lies in the fact that it lifted the entire theory of infinite series to a new level and brought new ideas and themes.

Still Euler was not satisfied, since he was far from an exact evaluation. Then suddenly, he had an idea which led him to the goal. In his paper that gave this new method for the solution of the explicit evaluation he writes excitedly at the beginning: *So much work has been done on the series* $\zeta(n)$ *that it seems hardly likely that anything new about them may still turn up.... I, too, in spite of repeated efforts, could achieve nothing more than approximate values for their sums.... Now, however, quite unexpectedly, I have found an elegant formula for* $\zeta(2)$*, depending on the quadrature of a circle [i.e., upon* π*]* (from Andre Weil's translation).

Euler's idea was based on an audacious generalization of Newton's formula for the sums of powers of the roots of a polynomial to the case when the polynomial was replaced by a power series. Writing a polynomial in the form

$$1 - \alpha s + \beta s^2 - \cdots \pm \rho s^k = \left(1 - \frac{s}{a}\right)\left(1 - \frac{s}{b}\right)\cdots\left(1 - \frac{s}{r}\right)$$

we have

$$\alpha = \frac{1}{a} + \frac{1}{b} + \cdots + \frac{1}{r}, \qquad \beta = \frac{1}{ab} + \frac{1}{ac} + \ldots$$

and so on. In particular

$$\frac{1}{a^2} + \frac{1}{b^2} + \cdots + \frac{1}{r^2} = \alpha^2 - 2\beta$$

and more generally

$$S_3 = \alpha^3 - 3\alpha\beta + 3\gamma, S_4 = \alpha^4 - 4\alpha^2\beta + 4\alpha\gamma + 2\beta^2 - 4\delta$$

and so on, where

$$S_k = \frac{1}{a^k} + \frac{1}{b^k} + \cdots + \frac{1}{r^k}.$$

Euler's idea was to apply these relations wholesale to the case *when the polynomial is replaced by a power series*

$$1 - \alpha s + \beta s^2 - \ldots,$$

indeed, to the power series

$$1 - \sin s = 1 - s + \frac{s^3}{6} - \ldots.$$

The function $1 - \sin s$ has the roots (all roots are double)

$$\frac{\pi}{2}, \frac{\pi}{2}, -\frac{3\pi}{2}, -\frac{3\pi}{2}, \frac{5\pi}{2}, \frac{5\pi}{2}, \ldots,$$

and so the above formulas give the following. First,

$$\frac{4}{\pi}\left(1-\frac{1}{3}+\frac{1}{5}-\dots\right)=1,$$

which is Leibniz's result. But now one can keep going and get

$$\frac{8}{\pi^2}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\dots\right)=1,$$

which leads at once to

$$\zeta(2)=\frac{\pi^2}{6}.$$

One can go on and on, which is what Euler did, calculating $\zeta(2k)$ up to $2k=12$. In particular

$$\zeta(4)=1+\frac{1}{2^4}+\frac{1}{3^4}+\dots=\frac{\pi^4}{90}.$$

The same method can be applied to $\sin s$ and leads to the same results.

Euler communicated these (and other) results to his friends (the Bernoullis in particular), and very soon everyone that mattered knew of Euler's sensational discoveries. He knew that his derivations were open to serious objections, many of which he himself was aware of. The most important of the objections were the following: (1) How can one be sure that $1-\sin s$ does not have other roots besides the ones written? (2) If $f(s)$ is any function to which this method is applied, $f(s)$ and $e^s f(s)$ both have the same roots and yet they should lead to different formulae. Nevertheless the numerical evaluations bolstered Euler's confidence, and he kept working to achieve a demonstration that would satisfy his critics. It took him about ten years, but he finally succeeded in obtaining the famous product formula for $\sin s$:

$$\frac{\sin x}{x}=\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right). \tag{4}$$

Once this formula is established, all the objections disappear, as he himself remarked.

The proof of (4) by Euler was beautiful and direct. He wrote

$$\frac{\sin x}{x}=\lim_{n\to\infty}\frac{\left(1+\frac{ix}{n}\right)^n-\left(1-\frac{ix}{n}\right)^n}{2ix}$$

and factorized explicitly the polynomials

$$q_n(x):=\frac{\left(1+\frac{ix}{n}\right)^n-\left(1-\frac{ix}{n}\right)^n}{2ix}$$

to get

$$q_n(x)=\prod_{k=1}^{p}\left(1-\frac{x^2}{n^2}\frac{1+\cos\frac{2k\pi}{n}}{1-\cos\frac{2k\pi}{n}}\right) \qquad (n=2p+1).$$

The formula (4) is obtained by letting n go to ∞ term by term in the product. As would be natural to expect, Euler does not comment on this passage to the limit; a modern rigorous argument would add just the observation that the passage to the limit termwise can be justified by *uniform convergence*, as can be seen from the easily established estimate

$$\left|\frac{1+\cos\frac{2k\pi}{n}}{1-\cos\frac{2k\pi}{n}}\right| \leq C\frac{x^2}{k^2\pi^2}$$

where C is an absolute constant. The method is applicable to a whole slew of trigonometric as well as hyperbolic functions and allowed Euler to reach all the formulae obtained earlier by his questionable use of Newton's theorem. Among these are

$$1-\frac{\sin s}{\sin\sigma} = \prod_{n=-\infty}^{\infty}\left(1-\frac{s}{2n\pi+\sigma}\right)\left(1-\frac{s}{2n\pi+\pi-\sigma}\right).$$

For convergence purposes this should be rewritten as

$$\begin{aligned}1-\frac{\sin s}{\sin\sigma} &= \left(1-\frac{s}{\sigma}\right)\prod_{n=1}^{\infty}\left(1-\frac{s}{2n\pi+\sigma}\right)\left(1+\frac{s}{2n\pi-\sigma}\right)\\ &\times \prod_{n=1}^{\infty}\left(1-\frac{s}{(2n-1)\pi-\sigma}\right)\left(1+\frac{s}{(2n-1)\pi+\sigma}\right).\end{aligned}$$

From the product formula (4) one can calculate by Newton's method the values of $\zeta(2k)$ explicitly; there are no problems (as everything in sight is absolutely convergent), and Euler did this. These evaluations, especially the value

$$\zeta(12) = \frac{691}{6825\times 93555}\,\pi^{12},$$

must have suggested to him that the Bernoulli numbers were lurking around the corner here, since

$$B_{12} = -\frac{691}{2730}.$$

Euler then succeeded in getting a closed formula for all the $\zeta(2k)$.

The main idea is to logarithmically differentiate (4) (as was also observed immediately by Nicholas Bernoulli) to get (replacing x by $s\pi$)

$$\pi\cot s\pi = \frac{1}{s}+\sum_{n=1}^{\infty}\left(\frac{1}{n+s}-\frac{1}{n-s}\right) \qquad (0<s<1). \tag{5}$$

The formula is written in such a way that absolute convergence is manifest; Euler did not bother with such niceties and wrote it as

$$\pi\cot s\pi = \sum_{-\infty}^{\infty}\frac{1}{s+n}. \tag{6}$$

It is definitely more convenient to do this, interpreting the sum as a *principal value*. We shall do so from now on, omitting the reference to principal values for brevity. Expressing the cotangent in terms of exponentials leads one to the function

$$B(s) = B(-s) := \frac{s}{e^s-1}-1+\frac{1}{2}s = \sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}s^{2k}.$$

The B_{2k} are the *Bernoulli numbers*, introduced by Jakob Bernoulli many years before Euler; Euler suggested they be called Bernoulli numbers. For the first few we have

$$B_2 = \frac{1}{6},\ B_4 = -\frac{1}{30},\ B_6 = \frac{1}{42},\ B_8 = -\frac{1}{30},\ B_{10} = \frac{5}{66},\ B_{12} = -\frac{691}{2730}.$$

Then

$$\pi \cot s\pi - \frac{1}{s} = \frac{2\pi i}{2\pi i s} B(2\pi i s) = 2\pi i \sum_{k=1}^{\infty} B_{2k} \frac{(2\pi i s)^{2k-1}}{(2k)!}.$$

Calculating derivatives at $s = 0$ we get Euler's surpassingly beautiful formula

$$\zeta(2k) = \frac{(-1)^{k-1} 2^{2k-1} B_{2k}}{(2k)!} \pi^{2k}. \tag{7}$$

Nowadays it is customary to treat s as a complex variable and establish (5) or (6) by complex methods, using periodicity and Liouville's theorem. I think however that Euler's method is unrivaled in its originality and directness. For a treatment of these formulae that is very close to Euler's and even more elementary in the sense that one works entirely over the real field, see Omar Hijab's very nice book [Hi]. One should also note that the results of Euler may be viewed as the forerunners of the work of Weierstrass and Jacobi, of infinite products with specified zeros and poles, with sums over lattices in the complex plane replacing sums over integers ($\wp$ and ϑ-functions).

In addition to the zeta values Euler also determined the values

$$L(2k+1) = 1 - \frac{1}{3^{2k+1}} + \frac{1}{5^{2k+1}} - \dots.$$

These are the very first examples of *twisting*, namely replacing a series by one where the coefficients are multiplied by a character mod N:

$$\sum_{n\geq 1} \frac{a_n}{n^s} \longmapsto \sum_{n\geq 1} \frac{a_n \chi(n)}{n^s}$$

where χ is a character mod N, more generally a function of period N. The transition from ζ to L corresponds to a character mod 4:

$$\chi(n) = \begin{cases} (-1)^{(n-1)/2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

I shall talk more about these when I discuss Euler products. The method for the sums $L(2k+1)$ is the same as for the zeta values and starts with the partial fraction

$$\frac{\pi}{\sin s\pi} = \sum_{-\infty}^{\infty} (-1)^n \frac{1}{s+n}$$

obtained by logarithmically differentiating the infinite product

$$1 - \frac{\sin x}{\sin s} = \prod_{n=-\infty}^{\infty} \left(1 - \frac{x}{2n\pi + s}\right)\left(1 - \frac{x}{2n\pi + \pi - s}\right)$$

at $x = 0$ and then changing s to $s\pi$.

It was natural for Euler to explore if the partial fraction expansions

$$\frac{\pi}{\sin s\pi} = \sum_{-\infty}^{\infty} (-1)^n \frac{1}{s+n}, \qquad \pi \cot s\pi = \sum_{-\infty}^{\infty} \frac{1}{s+n} \tag{8}$$

could be established by other methods. This he did in several beautiful papers, and his derivations take us through a whole collection of beautiful formulae in integral calculus, including the entire basic theory of what Legendre would later call the Eulerian integrals of the first and second kind, namely, the theory of the beta and gamma functions.

The starting point of the new derivation is the pair of formulae

$$(9)\qquad \begin{aligned} \int_0^1 \frac{x^{p-1}+x^{q-p-1}}{1+x^q}dx &= \sum_{-\infty}^{\infty}(-1)^n\frac{1}{p+nq} \quad (q>p>0) \\ \int_0^1 \frac{x^{p-1}-x^{q-p-1}}{1-x^q}dx &= \sum_{-\infty}^{\infty}\frac{1}{p+nq} \quad (q>p>0). \end{aligned}$$

These are derived by expanding

$$\frac{1}{1\pm x^q}$$

as power series and integrating term by term. One has to be a bit careful in the second of these formulae since the integrals do not converge separately. It is then a question of evaluating the integrals directly to obtain the formulae

$$(10)\qquad \begin{aligned} \int_0^1 \frac{x^{p-1}+x^{q-p-1}}{1+x^q}dx &= \frac{\pi}{q\sin(p/q)\pi} \quad (q>p>0), \\ \int_0^1 \frac{x^{p-1}-x^{q-p-1}}{1-x^q}dx &= \frac{\pi\cot(p/q)\pi}{q} \quad (q>p>0). \end{aligned}$$

We then obtain (7) with $s=p/q$. For Euler this was sufficient; we would add to his derivation a remark about justifying the continuity of both sides of the formulae in s.

For proving (10) Euler developed a method based on a beautiful generalization of the familiar formula (indefinite integration)

$$\int \frac{dx}{1+x^2} = \arctan x.$$

Euler obtains for

$$\int_0^x \frac{x^{m-1}}{1+x^{2n}}dx \qquad (2m>n>0. m, n \text{ integers})$$

the formula

$$\frac{(-1)^{m-1}}{2n}\sum_{k=1}^{n}\cos(2k-1)m\frac{\pi}{2n}\log\left(1+2x\cos(2k-1)\frac{\pi}{2n}+x^2\right)$$
$$+\frac{(-1)^{m-1}}{n}\sum_{k=1}^{n}\sin(2k-1)m\frac{\pi}{2n}\arctan\frac{x\sin(2k-1)\frac{\pi}{2n}}{1+x\cos(2k-1)\frac{\pi}{2n}}.$$

The formula is obtained using partial fractions and the factorization of $(1+x^{2n})$. We now let $x\to\infty$ in this formula. Using the identities (which Euler derived as special cases of a whole class of trigonometric identities)

$$\sum_{k=1}^{n} \cos \frac{(2k-1)m\pi}{2n} = 0$$

$$\sum_{k=1}^{n} (2k-1) \sin \frac{(2k-1)m\pi}{2n} = \frac{(-1)^{m-1}n}{\sin \frac{m\pi}{2n}},$$

we get, with Euler,

$$\int_0^\infty \frac{x^{m-1}}{1+x^{2n}} dx = \frac{\pi}{2n \sin \frac{m\pi}{2n}}.$$

We put $p = m, q = 2n$ and rewrite this as

$$\int_0^\infty \frac{x^{p-1}}{1+x^q} dx = \frac{\pi}{q \sin \frac{p\pi}{q}} \qquad (q > p > 0).$$

Here q is even; but if q is odd, the substitution $x = y^2$ changes the integral to one with the even integer $2q$, and we obtain the above formula for odd q also. Euler does not stop with this of course; he goes on to evaluate all the integrals of the form

$$\int_0^\infty \frac{x^{p-1}}{(1+x^q)^k} dx.$$

In particular he finds

$$\int_0^\infty \frac{x^{m-1}}{1 - 2x^n \cos\omega + x^{2n}} dx = \frac{\pi \sin \frac{n-m}{n}(\pi - \omega)}{n \sin\omega \sin \frac{(n-m)\pi}{n}}.$$

For $\omega = \frac{\pi}{2}$ this reduces to the previous formula.

The derivation of the second integral in (10) is similar but more complicated since we have to take into account the fact that the integrals do not converge separately. It is based on getting a formula for

$$\int_0^x \frac{x^{m-1}}{1-x^{2n}} dx$$

and we omit the details. The result is

$$\int_0^1 \frac{x^{m-1} - x^{2n-m-1}}{1-x^{2n}} dx = \frac{\pi}{2n} \cot \frac{m\pi}{2n},$$

which leads as before to the second formula in (10). It is to be noted that in this method also the factorization of $(1 \pm x^{2n})$ enters decisively, exactly as in his original proof of the infinite product for $\sin x$.

Finally one could also obtain (10) as a consequence of the theory of the gamma function, using only formulae that were known to Euler. We are used to writing

$$\Gamma(s+1) = \int_0^\infty e^{-x} x^s dx,$$

but Euler always preferred to write it as

$$[s] = s! = \int_0^1 (-\log x)^s dx$$

and think of it as an *interpolation* for n! He knew the functional equation

$$[s] = (s+1)[s-1]$$

as well as the formula

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin s\pi},$$

(the corollary)

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi},$$

and the limit formula

$$\Gamma(1+m) = \lim_{m\to\infty} \frac{1.2.\dots n}{(m+1)(m+2)\dots(m+n)}(n+1)^m,$$

which he would write as

$$[m] = \lim_{m\to\infty} \frac{1.2^m}{m+1}\frac{2^{1-m}.3^m}{m+2}\dots\frac{n^{1-m}(n+1)^m}{m+n}.$$

In fact it is in this form he introduces the Gamma function in one of his early letters to Goldbach. The derivation of (10) is now a straightforward consequence of the theory of these integrals. One gets

$$\int_0^1 \frac{x^{q-p-1}}{1+x^q}dx = \int_1^\infty \frac{x^{p-1}}{1+x^q}dx$$

so that

$$\int_0^1 \frac{x^{p-1}+x^{q-p-1}}{1+x^q}dx = \int_0^\infty \frac{x^{p-1}}{1+x^q}dx = \frac{1}{q}B\left(\frac{p}{q}, 1-\frac{p}{q}\right) = \frac{\pi}{q\sin(p/q)\pi}.$$

Once again the treatment of the second integral in (10) is more delicate.

The partial fractions (9) can be differentiated and specialized to yield explicit values for many infinite series. Euler worked out a whole host of these, with or without the twisting mentioned earlier. The sums he treated are of the form

$$\sum_{n\in\mathbf{Z}} \frac{h(n)}{(nq+p)^r}$$

where h is a periodic function, and their values are of the form

$$g\pi^r$$

where g is a *cyclotomic number*. The series he obtains are actually Dirichlet series corresponding to various characters mod q and their variants. Thus, with χ as the non-trivial character mod 3, extended to $\mathbf{Z}$ by 0,

$$\frac{2\pi}{3\sqrt{3}} = \sum_{n=1}^\infty \frac{(-1)^{n-1}\chi(n)}{n}, \qquad \frac{\pi}{3\sqrt{3}} = \sum_{n=1}^\infty \frac{\chi(n)}{n},$$

which he would write as

$$\frac{2\pi}{3\sqrt{3}} = 1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} + \frac{1}{8} - \frac{1}{10} - \frac{1}{11} + \dots$$

$$\frac{\pi}{3\sqrt{3}} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \dots.$$

Also

$$\frac{\pi^2}{8\sqrt{2}} = \sum_{n=1}^\infty \frac{\chi_8(n)}{n^2} \qquad \frac{\pi^2}{6\sqrt{3}} = \sum_{n=1}^\infty \frac{\chi_{12}(n)}{n^2}$$

where

$$\chi_8(n) = \begin{cases} +1 & \text{if } n \equiv \pm 1 \bmod 8 \\ -1 & \text{if } n \equiv \pm 3 \bmod 8 \\ 0 & \text{if otherwise} \end{cases} \qquad \chi_{12}(n) = \begin{cases} +1 & \text{if } n \equiv \pm 1 \bmod 12 \\ -1 & \text{if } n \equiv \pm 5 \bmod 12 \\ 0 & \text{if otherwise,} \end{cases}$$

which he would write as

$$\frac{\pi^2}{8\sqrt{2}} = 1 - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} + \dots$$

$$\frac{\pi^2}{6\sqrt{3}} = 1 - \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{11^2} + \dots$$

and so on.

Multizeta values. Throughout his life Euler tried to determine the zeta values at odd integers, $\zeta(3), \zeta(5), \dots$ but was unsuccessful. He obtained many formulae linking them but was unable to get a breakthrough. Late in his life, almost thirty years after his discoveries, he wrote a beautiful paper where he introduced what are now called *multizeta values*. The double zeta values are nowadays defined as

$$\zeta(a,b) = \sum_{m>n>0} \frac{1}{m^a n^b} \qquad (a, b \in \mathbf{Z}, a \geq 2, b \geq 1).$$

This is a slight variant of Euler's definition which we write as $\zeta_E(a,b)$, in which he would sum for $m \geq n$ and write the sum as

$$1 + \frac{1}{2^a}\left(1 + \frac{1}{2^b}\right) + \frac{1}{3^a}\left(1 + \frac{1}{2^b} + \frac{1}{3^b}\right) + \dots$$

so that

$$\zeta_E(a,b) = \zeta(a,b) + \zeta(a+b).$$

He proved the beautiful relation

$$\zeta(2,1) = \zeta(3)$$

as well as the more general

$$\zeta(p,1) + \zeta(p-1,2) + \dots + \zeta(2,p-1) = \zeta(p+1)$$

from which he derived the relations

$$2\zeta(p-1,1) = (p-1)\zeta(p) - \sum_{2 \leq q \leq p-2} \zeta(q)\zeta(p-q).$$

In recent years people have defined the multizeta values by

$$\zeta(s_1, s_2, \dots, s_r) = \sum_{n_1 > n_2 > \dots > n_r > 0} \frac{1}{n_1^{s_1} n_2^{s_2} \dots n_r^{s_r}} \quad (s_i \in \mathbf{Z}, s_1 \geq 2, s_i \geq 1).$$

The Euler identities have been generalized, new identities have been discovered by Ecalle and others, and considerable progress has been made about the nature of these numbers, including the odd zeta values. I mention the results that $\zeta(3)$ is irrational [A], that an infinity of the odd zeta values are irrational [BR], and that at least one of $\zeta(5), \zeta(7), \dots, \zeta(21)$ is irrational [R]. The multizeta values have been interpreted as *period integrals*, and this interpretation may possibly lead to a better understanding of them [KZ], [D1]. For more details and references see [V].