# Around Hall algebras in 23 slides

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# Hall Algebras

Let $\mathcal{C}$ be a small abelian category, such that

- $\mathrm{gldim}(\mathcal{C}) < \infty$, i.e. $\mathrm{Ext}^n(A, B) = 0$ for any $A, B \in Ob(\mathcal{C})$ and $n \gg 0$;
- $|\mathrm{Ext}^i(A, B)| < \infty$ for any $A, B \in Ob(\mathcal{C})$ and all $i \geq 0$.

**Definition.** *The* ***multiplicative Euler form*** $\langle \cdot, \cdot \rangle : K(\mathcal{C} \times \mathcal{C}) \to \mathbb{C}$ *is the form given by*

$$\langle A, B \rangle := \left(\prod_{i=0}^{\infty} |Ext^i(A, B)|^{(-1)^i}\right)^{1/2}.$$

Let $\mathcal{C}^{iso}$ be the set of isomorphism classes of objects in $\mathcal{C}$ and consider the vector space $\mathcal{H}(\mathcal{C}) := \bigoplus_{A \in \mathcal{C}^{iso}} \mathbb{C}[A]$. The following operation defines the structure of an associative algebra on $\mathcal{H}(\mathcal{C})$:

$$[A] \star [B] := \langle A, B \rangle \sum_C P^C_{A,B}[C],$$

where $\mathcal{P}^C_{A,B}$ is the number of short exact sequences (SES) $0 \to B \to C \to A \to 0$ and $P^C_{A,B} := \dfrac{\mathcal{P}^C_{A,B}}{|\mathrm{End}(A)||\mathrm{End}(B)|}$.

**Remark.** *The unit $i : \mathbb{C} \to \mathcal{H}(\mathcal{C})$ is given by $i(\lambda) = \lambda[0]$, where $0$ is the initial object of $\mathcal{C}$.*

**Example.** *Let $\mathcal{C}$ be the category of finite-dimensional vector spaces over a finite field $\Bbbk = \mathbb{F}_q$. The classes of objects in $\mathcal{C}^{iso}$ are given by $\{V_n\}_{n\geq 0}$ with $V_n := \Bbbk^n$. We notice that $Ext^{>0}(V_n, V_m) = 0$, while $|Ext^0(V_n, V_m)| = |Hom(V_n, V_m)| = q^{nm}$. Moreover, the number of SES*

$$0 \to V_n \to V_s \to V_m \to 0$$

*is zero, unless $s = m + n$. In case $s = m + n$ the number of SES as above, up to isomorphism of the first and third term, is $|Gr_{\Bbbk}(n, m+n)|$, where $Gr_{\Bbbk}(n, m+n)$ is the Grassmannian of $n$-dimensional subspaces in $(m+n)$-dimensional vector space. We conclude that $[V_m] \star [V_n] = q^{nm/2}\begin{bmatrix} n+m \\ n \end{bmatrix}_q V_{n+m}$, where $\begin{bmatrix} n+m \\ n \end{bmatrix}_q := \frac{[n+m]_q!}{[n]_q![m]_q!}$ with $[n]_q! := \dfrac{(q^n-1)(q^{n-1}-1)\dots(q-1)}{(q-1)^n}$ is the $q$-binomial coefficient. It is equal to the number of points on $Gr_{\Bbbk}(n, m+n)$.*

**Remark.** *Notice that similar to the binomial coefficients, their 'q-analogs' satisfy the equality* $\begin{bmatrix} n+m \\ n \end{bmatrix}_q = \begin{bmatrix} n+m \\ m \end{bmatrix}_q$. *It follows that* $[V_m] \star [V_n] = [V_n] \star [V_m]$ *and the Hall algebra* $\mathcal{H}(\mathcal{C})$ *is commutative. Moreover, the algebra* $\mathcal{H}(\mathcal{C})$ *is generated by* $[V_1]$ *and isomorphic to the ring of polynomials in one variable* $\mathbb{C}[x]$. *It is straightforward to check that* $[V_1]^{\star n} = \sqrt{q^{\frac{n(n-1)}{2}}}[n]_q![V_n]$ *(notice that* $[n]_q!$ *is the number of points on the variety of complete flags over* $\mathbb{F}_q$ *and* $q^{\frac{n(n-1)}{2}} = q^{1\cdot 1}q^{1\cdot 2}\dots q^{1\cdot(n-1)}$*). Hence, the isomorphism of algebras* $\varphi : \mathbb{C}[x] \xrightarrow{\sim} \mathcal{H}(\mathcal{C})$ *with* $\varphi(x) = [V_1]$ *has* $\varphi(x^n) = q^{\frac{n(n-1)}{4}}[n]_q![V_n]$.

# Quivers

**Definition.** *A* ***quiver*** $Q = (Q_0, Q_1)$ *is a finite directed graph with finitely many vertices enumerated by the set* $Q_0$ *and finitely many edges indexed by* $Q_1$*. Each edge is uniquely determined by the pair of vertices it connects, which we will denote by* $t(a)$ *and* $h(a)$ *standing for 'tail' and 'head', respectively. A* ***representation of a quiver*** $Q$ *consists of a collection of vector spaces* $\{V_i\}_{i \in Q_0}$ *and linear homomorphisms* $\alpha_a \in Hom(V_{ta}, V_{ha})$ *for each arrow* $a \in Q_1$.

Such representations form a category with morphisms being collections of $\mathbb{C}$-linear maps $\psi_i : V_i \to W_i$ for all $i \in Q_0$ such that the diagrams

$$\begin{array}{ccc} V_{ta} & \xrightarrow{\alpha_a} & V_{ha} \\ \Big\downarrow{\scriptstyle \psi_{ta}} & & \Big\downarrow{\scriptstyle \psi_{ha}} \\ W_{ta} & \xrightarrow{\alpha'_a} & W_{ha} \end{array}$$

commute.

This category will be denoted by $\mathrm{Rep}(Q)$. There is a natural way to associate a Kac-Moody Lie algebra $\mathfrak{g}_Q$ to $Q$. Namely, the Cartan matrix for $\mathfrak{g}_Q$ is $C = 2 \cdot I - A_Q - A_Q^T$, where $A_Q$ is the adjacency matrix of $Q$.

**Definition.** *A* ***path*** *$p$ in a quiver $Q = (Q_0, Q_1)$ is a sequence $a_\ell a_{\ell-1} \dots a_1$ of arrows in $Q_1$ such that $t(a_{i+1}) = h(a_i)$ for $i = 1, 2, \dots, \ell - 1$. In addition, for every vertex $x \in Q_0$ we introduce a path $e_x$.*

*The* ***path algebra*** *$\mathcal{P}_Q$ is a $\Bbbk$-algebra with a basis labeled by all paths in $Q$. The multiplication in $\mathcal{P}_Q$ is given by*

$$p \cdot q := \begin{cases} pq, & \text{if } t(p) = h(q) \\ 0, & \text{otherwise,} \end{cases}$$

*where $pq$ stands for the concatenation of paths subject to the conventions that $pe_x = p$ if $t(p) = x$, and $e_x p = p$ if $h(p) = x$.*

**Remark.** *Notice that $\mathcal{P}_Q$ is of finite dimension over $\Bbbk$ if and only if $Q$ has no oriented cycles. The path algebra has a natural grading by path length with the subring of grade zero spanned by the trivial paths $e_x$ for $x \in Q_0$. It is a semisimple ring, in which the elements $e_x$ are orthogonal idempotents.*

**Theorem.** *The category $Rep(Q)$ is equivalent to the category of finitely-generated left $\mathcal{P}_Q$-modules. In particular, $Rep(Q)$ is an abelian category.*

**Remark.** *If $Q$ has no oriented cycles, then the category $Rep(Q)$ is hereditary, i.e. $Ext^i(A, B) = 0$ for any $i \geq 2$ and $A, B \in Rep(Q)$.*

**Example.** *Let $Q$ be the quiver $\bullet_1 \longrightarrow \bullet_2$. An object in $Rep(Q)$ is a pair of vector spaces $(V_1, V_2)$ together with a linear map $a \in Hom(V_1, V_2)$. There are two simple objects $S_1 : \Bbbk \to 0$ and $S_2 : 0 \to \Bbbk$ and one (up to isomorphism) indecomposable, which is not simple $I_{12} = \Bbbk \xrightarrow{id} \Bbbk$. The adjacency matrix $A_Q = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ gives rise to Cartan matrix $C = 2 \cdot I - A_Q - A_Q^T = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ corresponding to Lie algebra $\mathfrak{sl}_3$ of traceless $3 \times 3$ matrices. The path algebra, $\mathcal{P}(Q)$, is of dimension 3 over $\Bbbk$. It is generated by two idempotents $e_1, e_2$ and an element $a$ subject to relations $ae_1 = e_2 a = a$, $e_1^2 = e_1, e_2^2 = e_2$ and $ae_2 = e_1 a = a^2 = 0$.*

**Remark.** *Notice that we have a natural bijection between simple roots $E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and simple objects in $Rep(Q)$.*

Let $U_q(\mathfrak{g})$ be the quantized enveloping algebra with $\mathfrak{g}$ the Lie algebra associated to the Dynkin diagram formed by $Q$. We denote the simple roots of $\mathfrak{g}$ by $E_i$ and simple representations of $Q$ by $\{S_i\}_{i\in Q_0}$.

The following result was obtained by Ringel and Green.

**Theorem.** *Let $\Bbbk$ be a finite field and $v = \sqrt{|\Bbbk|}$. There is an embedding of algebras $\varphi : U_v(\mathfrak{n}_+) \hookrightarrow \mathcal{H}(Rep_{\Bbbk}(Q))$ with $\varphi(E_i) = [S_i]$ (here $\mathfrak{n}_+$ is the standard maximal nilpotent subalgebra in $\mathfrak{g}$).*

Let $\mathcal{C} = R\text{-}mod$ be a category of finite–dimensional left modules over a fixed finite–dimensional, associative $\mathbb{C}$–algebra $R$. There is a way to associate a Hall algebra $\mathcal{H}(\mathcal{C})$ to $\mathcal{C}$. The construction was sketched by Kapranov and Vasserot and later given in detail by Joyce. Notice that if $Q$ has no oriented cycles, then its path algebra $\mathcal{P}(Q)$ has the required property, therefore, we can associate a Hall algebra to the category of finite–dimensional left modules over this algebra. The latter is equivalent to $Rep(Q)$.

**Theorem.** *There is an embedding of algebras $\varphi : U(\mathfrak{n}_+) \hookrightarrow \mathcal{H}(Rep_{\mathbb{C}}(Q))$.*

# McKay correspondence

Let $G \subset GL_n(\mathbb{C})$ be a finite subgroup and consider the affine variety $X = \mathbb{C}^n/G := Spec(\mathbb{C}[x_1, x_2, \dots, x_n])^G$. We are interested in examples with the following properties

1. $X$ has an isolated singularity at 0;
2. there exists a projective resolution $\pi : Y \to X$
3. there is a bijection

$$\{\text{irr. comp. of } \pi^{-1}(0)\} \overset{1:1}{\longleftrightarrow} \{\rho \in \text{Irr}(G) \setminus \text{triv}\}$$

A good candidate for such a resolution $Y$ is the $G$-Hilbert scheme $G\text{-Hilb}(\mathbb{C}^n)$.

**Definition.** *A **cluster** $\mathcal{Z} \subset \mathbb{C}^n$ is a zero-dimensional subscheme and a **G-cluster** is a $G$-invariant cluster, s.t. $H^0(\mathcal{O}_{\mathcal{Z}}) \simeq \mathcal{R}$ (the regular representation of $G$). The **G-Hilbert scheme** ($G$-Hilb($\mathbb{C}^n$)) is the fine moduli space parameterizing $G$-clusters.*

**Example.** *Let $G = \mathbb{Z}_r$ be embedded into $SL_2(\mathbb{C})$ via*

$$\varphi : \mathbb{Z}_r \hookrightarrow SL_2(\mathbb{C}),\ \varphi(1) = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \text{ with } \zeta = e^{\frac{2\pi i}{r}}.$$

*Then $X = Spec(\mathbb{C}[x,y])^G \simeq \mathbb{C}[u,v,w]/(uv - w^r)$ with $u = x^r, y = v^r, w = xy$.*

Using the definition of $G$-Hilb, we get

$$Y := G\text{-}\mathrm{Hilb}(\mathbb{C}^2) = \{I_{\mathcal{Z}} \subset \mathbb{C}[x,y] \mid H^0(\mathcal{O}(\mathcal{Z})) = \mathbb{C}[x,y]/I_{\mathcal{Z}} \simeq \mathcal{R}\},$$

where $\mathcal{R} \simeq \bigoplus_{i=0}^{n} \rho_i$ for $\rho_i : \mathbb{Z}_r \to \mathbb{C}^*$, $\rho_i(1) = \zeta^i$.

**Fact.** *$Y$ is smooth and the map $\pi : Y \to X$ given by $\pi(I_{\mathcal{Z}}) = supp(I_{\mathcal{Z}})$ is a projective resolution. Moreover, $X$ has an isolated singularity at the origin. The central fiber is*

$$\pi^{-1}(0) = \bigcup_{j=1}^{r-1} I_{\lambda_j,\mu_j}$$

*with* $I_{\lambda_j,\mu_j} = \langle \lambda_j x^j - \mu_j y^{r-j}, xy, x^{j+1} \rangle \simeq \mathbb{P}^1 = [\lambda_j : \mu_j]$.

**Remark.** $I_{\lambda_j,\mu_j} \cap I_{\lambda_k,\mu_k} = \begin{cases} pt, & |k-j| = 1 \\ \varnothing & otherwise. \end{cases}$


$\pi^{-1}(0)$, type $A_4$ Kleinian singularity

dual

Dynkin diagram $A_4$

**Fact.** *Let $G \subset SL_2(\mathbb{C})$ be a finite subgroup, $\pi : G\text{-}Hilb(\mathbb{C}^2) \to X$ the crepant projective resolution. Then*

1. *the number of irreducible components $(E_i)$ of $\pi^{-1}(0)$ coincides with the number of nontrivial irreps of $G$;*
2. *the graph, dual to the intersection graph of $E_i$'s is the Dynkin diagram (subgraph of McKay quiver $Q = (G, V)$), in particular, the Cartan matrix is the negative of the intersection matrix (with entries $E_{ij} := E_i \cdot E_j$):*

**Example.** *Type A*

$$C_n = \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix} \text{ and } E_n = \begin{pmatrix} -2 & 1 & 0 & 0 & \dots & 0 \\ 1 & -2 & 1 & 0 & \dots & 0 \\ 0 & 1 & -2 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & -2 & 1 \\ 0 & 0 & \dots & 0 & 1 & -2 \end{pmatrix}$$

# McKay Quiver

**Definition.** *Let $G \subset GL(V)$ be a finite abelian subgroup. The **McKay quiver** $Q(G,V)$ is the graph given by the following*

$$\{vertices\ of\ Q\} \xleftrightarrow{1:1} \{irreps\ of\ G\}$$

$$\#\{edges\ i \to j\} = dim(Hom_G(\rho_i \otimes V, \rho_j))$$

*A **representation** of $Q$ is an additional collection of data: assign a vector space $V_\rho$ of dimension $dim(\rho) = 1$ to every vertex (according to the irrep $\rho$ it is associated to) and a linear map (number) $x_{ij} \in Hom(V_{\rho_i}, V_{\rho_j})$ to every edge $i \to j$ subject to the relations*

$$\langle x_{jk}x_{ij} = x_{kj}x_{ik} \rangle.$$

**Remark.** *Representations of $G$ are one-dimensional and correspond to characters of $G$:*

$$char(G) := \{\chi : G \to \mathbb{C}^*\}.$$

*In particular, as a representation of $G$, we have $\mathbb{C}^n = \bigoplus_{i=1}^{n} \mathbb{C}\chi_i =: \bigoplus_{i=1}^{n} \mathbb{C}e_i$ and let $x_1, x_2, \ldots, x_n \in (\mathbb{C}^n)^*$ be the dual basis to $\{e_1, e_2, \ldots, e_n\}$ with $R = \mathbb{C}[x_1, x_2, \ldots, x_n]$ the coordinate ring of $\mathbb{C}^n$. The chain of isomorphisms $Hom_G(\chi_k \otimes \mathbb{C}^n, \chi_\ell) \simeq Hom_G(\chi_k \otimes \bigoplus_{i=1}^{n} \mathbb{C}e_i, \chi_\ell) \simeq \bigoplus_{i=1}^{n} Hom_G(\chi_k \otimes \mathbb{C}e_i, \chi_\ell)$ provides a natural identification of the maps assigned to the arrows in the McKay quiver $Q(G, \mathbb{C}^n)$ with multiplication by $x_i$'s and, hence, impose the relations corresponding to commutation of the latter that we imposed on the previous slide.*

# Modern formulation of McKay correspondence

Let $Coh_G(\mathbb{C}^n)$ be the category of $G$-equivariant coherent sheaves on $\mathbb{C}^n$, and $Coh(Y)$ be the category of coherent sheaves on $Y$. The McKay correspondence is the derived equivalence

$$\Psi : D^b(Coh_G(\mathbb{C}^n)) \to D^b(Coh(Y))$$

Any finite-dimensional representation $V$ of $G$ gives rise to two equivariant sheaves on $\mathbb{C}^n$: the skyscraper sheaf $V^0 = V \otimes_{\mathbb{C}} \mathcal{O}_0$, whose fiber at 0 is $V$ and all the other fibers vanish, and the locally free sheaf $\widetilde{V} = V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n}$.

**Remark.** *There is an equivalence of abelian categories*

$$\Theta : Rep(Q(G, \mathbb{C}^n), \mathcal{R}) \simeq Coh_G(\mathbb{C}^n).$$

# Known results

The McKay correspondence holds in the following cases:

1. $G \subset SL_2(\mathbb{C})$, any $G$ (KV '98)
2. $G \subset SL_3(\mathbb{C})$, any $G$, $Y = G - Hilb(\mathbb{C}^3)$ (BKR '01)
3. $G \subset SL_3(\mathbb{C})$, any abelian $G$ (CI '04)
4. $G \subset SP_{2n}(\mathbb{C})$, $Y$ is a crepant symplectic resolution (BK '04)
5. $G \subset SL_n(\mathbb{C})$, any abelian $G$, $Y$ is a projective crepant symplectic resolution (Kawamata)

A natural question: what are the images of $\widetilde{\rho}$ and $\rho^0$ ($\rho \in \mathrm{Irr}(G) \setminus triv$) under the equivalence?

1. $\Psi(\widetilde{\rho})$ is a vector bundle of dimension $\dim(\rho)$ and is called a tautological or GSp-V sheaf (after Gonzales-Sprinberg and Verdier).

2. Relatively little is known about $\Psi(\rho^0)$.

The following results are due to Kapranov, Vasserot and Logvinenko.

**Theorem.** 1. *Let $G \subset SL_2(\mathbb{C})$ be a finite subgroup and $\rho \in Irr(G) \setminus triv$. Then $\Psi(\rho^0) \simeq \mathcal{O}_{\mathbb{P}^1_\rho}(-1)[1]$.*

2. *Let $G \subset SL_3(\mathbb{C})$ be a finite abelian subgroup, s. t. $X = \mathbb{C}^3/G$ has an isolated singularity at the origin. Then for any $\rho \in Irr(G) \setminus triv$, the object $\Psi(\rho^0) \in D^b(Coh(Y))$ is pure (here $Y = G\text{-}Hilb(\mathbb{C}^3)$ and an object is called **pure** provided all cohomology groups, except one, vanish).*

**Remark.** *The KV result gives a natural way to associate nontrivial irreps with irreducible components of the central fiber (this is consistent with the correspondence that we established earlier).*

Suppose $G \subset SL_3(\mathbb{C})$ satisfies the following assumptions:

1. the McKay quiver $Q(G, \mathbb{C}^3)$ contains a subquiver $Q'$ (without oriented cycles) with $\mathcal{R} \cap \mathcal{P}_{Q'} = 0$;
2. $\Psi$ sends the skyscraper sheaves $\chi^! \in Coh_G(\mathbb{C}^n)$, corresponding to the simple representations in $\mathrm{Rep}(Q(G, \mathbb{C}^3), \mathcal{R})$ supported at the vertices of $Q'$, to pure sheaves concentrated in the same degree.

Kapranov and Vasserot have also observed that if $\mathcal{C}_1$ and $\mathcal{C}_2$ are $\mathbb{C}$-linear finitary abelian categories, there is a derived equivalence $\Psi : D^b(\mathcal{C}_1) \to D^b(\mathcal{C}_2)$ and a collection of objects $\{a_1, \ldots, a_n\}$ in $\mathcal{C}_1$, s.t. $\Psi(a_i)$ are all pure and concentrated in the same degree, then the Hall algebra generated by the objects $\{a_1, \ldots, a_n\}$ is isomorphic to the Hall algebra generated by their images $\{\Psi(a_1), \ldots, \Psi(a_n)\}$.

Let $\mathcal{H}\langle\{\Psi(\chi_i^!)\}_{i\in Q_0'}\rangle$ be the Hall algebra generated by the images of sheaves corresponding to simple representations of $Q'$ under $\Psi$ and $\mathfrak{n}_+ \subset \mathfrak{g}_{Q'}$ stand for the corresponding nilpotent subalgebra of $\mathfrak{g}_{Q'}$. It follows from the discussion above that one has an isomorphism of algebras:

$$\Theta : U(\mathfrak{n}_+) \to \mathcal{H}\langle\{\Psi(\chi_i^!)\}_{i\in Q_0'}\rangle.$$

In [2] I have found an infinite collection of cyclic finite abelian subgroups of $SL_3(\mathbb{C})$ satisfying the aforementioned conditions for each simply laced Dynkin diagram $Q'$ of affine type except $\widetilde{D}_4$. If you would like to learn more about Hall algebras, [1] is an excellent place to start!

# References


[1] O. Schiffmann, *Lectures on Hall algebras*, Geometric methods in representation theory. II, Sémin. Congr., vol. 24, Soc. Math. France, Paris, 2012, pp. 1–141 (English, with English and French summaries).

[2] B. Tsvelikhovskiy, *The universe inside Hall algebras of coherent sheaves on toric resolutions*, arXiv:2201.07847v2 (2022).