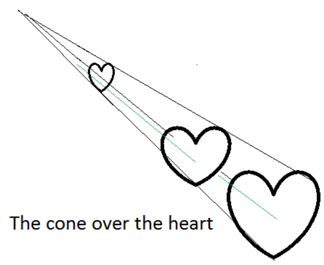
A proof of the Freudenthal's Suspension theorem using elementary techniques from Differential Topology [1]

Given a topological space X, to get its Cone, pile up copies of it, however, as you go up, shrink the sizes of your copies, until it collapses into a point (in finite time):



If we do the cone in the opposite direction as well, the space we get, the two cones glued on baases, is called the Suspension of the space X, denoted by ΣX.

One nice thing about ΣX is that if $X = S^k$, the $k-$ dimensional sphere, then $\Sigma X = S^{k+1}$, the sphere of 1 higher dimension. Here equality means up to homotopy.

If $f : X \longrightarrow Y$ is a continuous map, then by replicating it on copies of X and Y at each level, we get a map $\Sigma f : \Sigma X \longrightarrow \Sigma Y$.

Homotopy groups of spheres, that is the homotopy types of maps $f : S^i \longrightarrow S^n$, are very difficult to find. For instance, how can we know all the maps $S^4 \longrightarrow S^3$?

The construction above is telling us that some of the maps $S^4 \longrightarrow S^3$ come from the suspension of the maps $S^3 \longrightarrow S^2$. Therefore, knowing the suspension map can provide some insight into understanding these homotopy groups.

In 1937 Hans Freudenthal proved the following theorem:

THEOREM: Suspension
$$\Sigma : \pi_i(S^n) \longrightarrow \pi_{i+1}(S^{n+1})$$
is an isomorphism if $i < 2n - 1$. For $i = 2n - 1$, it is a surjection.

This theorem alone is the cornerstone of the stable homotopy theory. It is elegant, it is deep.

[1]Key Words: Suspension Map, homotopy groups of spheres, Isotopy

One usually sees this theorem as a corollary to a quite more general result. Wikipedia has a 1-paragraph proof, and claims that it is a corollary of the "homotopy Excision Theorem." Such proofs, although slick, are disappointing to me personally! They overshadow the immediate phenomena that are unfolding. The fact that after all you are working with spheres and maps between them is camouflaged. It is a gift wrapped into too much paper!

An alternative proof comes from framed cobordism. Our proof runs quite parallel to it.

I will present a proof that uses no machinery beyond the kind of differential topology found in the first chapters of Milnor's *Topology From The Differentiable Viewpoint.* Hopefully, one will see more of the behind-the-scenes than in other proofs.

Even if you're not into homotopy really, come for some surprising by-products! It is always good to have one or two technical tricks up your sleeve.

Behnam