ON THE MATTER OF METRIC SPACES

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Abstract. One of the biggest challenges in the mathematics of the general theory of relativity is that the tools of global analysis and the calculus of variations are ill-suited to the hyperbolic signature of space-time. For example, the time-like geodesics are "local maximizers" of the kinetic energy, in contrast to the case of Euclidean signature, where the geodesics are "local minimizers". In fact, in space-time there is only a viable Morse theory under very stringent global conditions (global hyperbolicity), as described in 1970s work of Uhlenbeck and others, and these global conditions rule out any kind of reasonable compactness. The considerations of this talk were due, in part, to the author's attempt to extend and describe a kind of Morse theory that is more suited to Lorentzian manifolds. The problem of how to compactify is non-trivial, dating back to early work of Geroch, Kronheimer, and Penrose, and still is not completely settled. Fields in space-time, like solutions of the wave equation, present still further challenges. Energy estimates do not lead in a geometrically natural way to function spaces, like the Sobolev spaces of the Euclidean theory.

This talk focuses specifically on a small part of this research programme, that is nonetheless very interesting and novel. We observe that it is possible to associate a certain metric space to any Lorentzian manifold (or, more generally, to any causal manifold in the sense of the author's 2015 University of Pittsburgh Dissertation). The appearance of a "natural" metric space associated to space-time is an encouraging sign that it should be possible eventually to bring general relativity into the conventional world of global mathematical analysis.

We shall discuss the geometry of this (sub-Riemannian) metric space. This geometry completely describes the timelike geodesics in the space-time, in the sense that any timelike geodesic is associated with a "zitterbewegung" of a geodesic in the metric space. In flat space, the geodesics of the metric space form a completely integrable Hamiltonian system, whose solutions can (in principle) be given explicitly in terms of elliptic functions. We shall also present some numerical simulations of these geodesics in two and three dimensions.

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