UNIQUENESS OF RELAXATION OSCILLATIONS: A CLASSICAL APPROACH

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Abstract. In a recent paper in this Journal, [2], the authors discuss a relaxation oscillator which apparently has not been put in standard Lienard form. They use geometric perturbation theory to analyze this model. Their main result is the existence and uniqueness of a periodic solution for small values of two parameters, δ and ε, and the behavior of this solution as δ and ε tend to zero. We show how standard ode methods can be used to give shorter and more direct proofs of these results. Along the way we give a new proof of a more general result.

1. Introduction

Nonlinear systems of two differential equations with unique periodic solutions have been widely studied. One of the earliest and most important of these is van der Pol's equation, which may be written in system form as

$$\begin{aligned} u' &= -\varepsilon v \\ v' &= u - \left(\frac{v^3}{3} - v\right). \end{aligned} \tag{1.1}$$

It is well known that for any $\varepsilon > 0$ there is a unique nontrivial periodic solution. More generally, much work has been done on what are called Lienard equations, which we may write as

$$\begin{aligned} u' &= -f(v) \\ v' &= u - g(v). \end{aligned}$$

Starting with Lienard, various authors gave conditions on f and g which ensure that this system also has a unique periodic solution. See [6] for a discussion and references. All the proofs of uniqueness which we have seen for systems in this generality have used "energy" functions of some kind, perhaps in complicated ways, which means that considerable ingenuity may be necessary to use these methods for more general systems.

In this paper we will consider the case of "relaxation oscillations", which are seen in (1.1) when ε is small. Such systems have a parameter, say ε, such that if $\varepsilon = 0$ then one of the equations is degenerate in some way. We will not attempt to give a general definition of relaxation oscillations. The motivation for this paper was the work [2], where a system which to our knowledge has not been put in Lienard form was studied using the methods of geometric perturbation theory. This system was considered earlier, for example in [7], as a model of glycolytic oscillations.

The geometry of the phase plane for this model is similar to that of (1.1). This means that it is easy to give a geometric argument to show the existence of a periodic solution, assuming only that the equilibrium point is unstable. However proving uniqueness and stability of this solution is more challenging.

Our first result, Theorem 1, is for a class of relaxation oscillators which includes the one studied in ([2]). The result is probably not new; see Remark 2 below for some citations of related work. However our proof of uniqueness and stability is simpler than others we have seen, because we don't require either geometric perturbation theory or asymptotic expansions around certain turning points in the problem as described below.

In the case studied in [2] there is a second small parameter, "δ". Following the proof of Theorem 1 we have a short section showing that it applies to the model of [2] for δ in the specific range $(0, \frac{1}{\sqrt{8}})$. For each δ in this range there is a unique periodic solution for sufficiently small ε. But more interestingly, in Theorem 3.1 the authors of [2] have been able to show that the range of existence and uniqueness in the (ε, δ) plane includes a region of the form $0 < \varepsilon < \tilde{\varepsilon}_0 \delta$, for small δ, thus estimating ε in terms of δ.

The methods used in [2] require a detailed examination of the three dimensional system obtained by adding δ as a third dependent variable, with its own differential equation, $\delta' = 0$. The result then depends on several rescaling regimes and "blow-up" methods of some complexity. The principle goal of this paper is to give a shorter and more direct proof of the main result in [2], Theorem 3.1, using standard ode methods in the plane.[1] Theorem 2 below restates the result in slightly different terminology, and its proof occupies the last half of this paper.

2. A general result

The equations we consider are of the form

$$a' = \varepsilon A(a,b)$$
$$b' = B(a,b). \tag{2.1}$$

We make the following five assumptions on A and B.

- **C1:** These functions are continuous with continuous second partial derivatives in a rectangle $R = \{(a,b) \mid c_1 \le a \le c_2,\ d_1 \le b \le d_2\}$.
- **C2:** The set $\Gamma = \{(a,b) \text{ in } R \mid B(a,b) = 0\}$ is an S-shaped curve with two turning points. More precisely, this set is the graph of a continuous function $a = \gamma(b)$ defined on $[d_1, d_2]$ with $\gamma(d_1) = c_1$ and $\gamma(d_2) = c_2$. The function γ has exactly one local maximum, at a point (b_1, a_2), and one local minimum, at (b_2, a_1), where $c_1 < a_1 < a_2 < c_2$ and $d_1 < b_1 < b_2 < d_2$. (See Figure 1.)
- **C3:** $\frac{\partial A}{\partial a} \le 0$, $\frac{\partial A}{\partial b} < 0$, and $\frac{\partial B}{\partial a} > 0$ in the interior of R.
- **C4:** Let Γ_1, Γ_2, and Γ_3 be the "branches" of Γ in $[d_1, b_1]$, $[b_1, b_2]$, and $[b_2, d_2]$, respectively. Then:
  - (i) $\frac{\partial B}{\partial b} < 0$ on Γ_1 and Γ_3 and $\frac{\partial B}{\partial b} > 0$ on Γ_2, except at the turning points
  - (ii) $\frac{\partial^2 B}{\partial b^2}(a_2, b_1) > 0$ and $\frac{\partial^2 B}{\partial b^2}(a_1, b_2) < 0$.

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[1]We thank a referee for calling our attention to an omission in what we originally proved, and for some helpful citations and other comments.

Conditions C1, C3 and C4 imply that the function γ in C2 has continuous first and second derivatives which satisfy

$$\begin{gathered}\gamma' > 0 \text{ in } [d_1, b_1),\ \gamma' < 0 \text{ in } (b_1, b_2), \\ \gamma' > 0 \text{ in } (b_2, d_2],\ \gamma''(b_1) < 0, \text{ and } \gamma''(b_2) > 0.\end{gathered} \tag{2.2}$$

C5: The system (2.1) has a unique equilibrium point in R. This equilibrium point, say (a^*, b^*), is on the interior of Γ_2.

Conditions C2 and C3 imply that on a vertical line in R which crosses the interior of all three branches, B is positive below Γ_1 and changes sign at each crossing of Γ.

Theorem 1. *Under conditions C1-C5 there is an $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, then the system (2.1) has a unique periodic solution in R. This solution is asymptotically orbitally stable. As $\varepsilon \to 0$ the orbit of the periodic solution approaches the piecewise smooth curve, or "singular solution", shown below in Figure 1.*

The proof of this result will be given in the following sections.

Remark 1. *These conditions are a generalization of the examples (1.1) and (3.1). The latter, analyzed in section 3, was studied in* [2]*, and in the proof that follows the reader may find it helpful to consider this particular case.*

Remark 2. *A result of this type appears in* [5]*. The proof there is complicated because it requires use of matched asymptotic expansions. In* [2] *a result like Theorem 1 for the case of the system (3.1) is stated, and for proof an earlier paper,* [3] *is cited. The proof in* [3] *is very short but relies heavily on Theorem 2.1 in* [4]*. The proof of that result requires a substantial amount of geometric perturbation theory and takes 12 pages. We are not aware of a proof of a result like Theorem 1 which is as direct and short as ours.*

We consider the initial value problem consisting of (2.1) with initial conditions

$$a(0) = a^*, b(0) = \beta \tag{2.3}$$

where a^* is introduced in condition C5, and from now on we will assume that (a^*, β) is in R and below Γ_1. For such a β, let

$$\mathbf{p}_\beta(t) = (a(t), b(t)) = (a(t,\beta), b(t,\beta))$$

be the unique solution of (2.1) and (2.3). Standard geometric arguments (briefly outlined in Section 2.1 below) ensure that for sufficiently small $\varepsilon > 0$ there is at least one β_1 such that $\mathbf{p}_{\beta_1}$ is periodic.[2] Also, if (a^*, β^*) is on Γ_1, then $\beta_1 < \beta^*$, and if $\bar\beta$ is the smallest such β_1 then $\left|\bar\beta - \beta^*\right| \to 0$ as $\varepsilon \to 0$. Our goal then is to prove in Section 2.2 that for sufficiently small ε there is a unique β_1 such that $\mathbf{p}_{\beta_1}$ is periodic.

2.1. **Proof of existence of a periodic solution, and its limiting orbit as $\varepsilon \to 0$.** The proof that for a fixed sufficiently small ε there is at least one solution is readily constructed by considering Figure 1, consulting standard references such as [5] if necessary. To discuss the limiting behavior of all periodic solutions, an argument is needed to exclude the existence of a small periodic orbit around (a^*, b^*). This is provided by the following lemma, the proof of which will also be useful when we consider the system in [2].

[2]For van der Pol's equation, R can be as large as desired, which implies that existence of a periodic solution can be proved for any $\varepsilon \in (0, \varepsilon_1)$, where ε_1 is given in C5. But this, and uniqueness, are well known in this case.

Lemma 1. *For sufficiently small* ε, (a^*, b^*) *is an unstable node. The matrix*

$$M = \begin{pmatrix} \varepsilon A_a & \varepsilon A_b \\ B_a & B_b \end{pmatrix} |_{(a^*,b^*)}$$

has eigenvalues $\lambda_2 > \lambda_1 > 0$ *where* $\lambda_1 = O(\varepsilon)$ *and* $\lambda_2 = B_b(a^*, b^*) + O(\varepsilon)$ *as* $\varepsilon \to 0$. *An eigenvector* $\mathbf{v} = (v_1, v_2)$ *corresponding to* λ_2 *has slope* $m_2 \leq -\frac{c}{\varepsilon}$ *for some* $c > 0$ *independent of* ε. *Assume that* $v_2 < 0$ *and let* Δ_ε *denote the unique trajectory of (2.1) which is tangent to* $\mathbf{v}$ *at* (a^*, b^*). *Then for sufficiently small* ε, *solutions* (a, b) *on* Δ_ε *have* $a' > 0$, $b' < 0$ *as long as* $b \geq b_1$ *and continue with* $a' > 0$, $b' < 0$ *until they cross the branch* Γ_1 *of* Γ.

The trajectory Δ_ε is pictured for two values of ε in Figure 1. The main content of this lemma is that for small ε this trajectory continues downward to cross Γ_1, (the red trajectory in Figure 1), instead of bending right to cross Γ_2 (the green trajectory).

Proof. The claims in the first three sentences result from routine calculations using the properties C3 and C4 (i). To verify the last sentence we first choose an ε_1 small enough that (a^*, b^*) is an unstable node. The eigenvectors necessarily have negative slope, and the one corresponding to λ_2 has the steepest slope. However if ε_1 is not sufficiently small, the trajectory Δ_{ε_1} will first cross Γ_2 rather than Γ_1.

Note that

$$\frac{\partial}{\partial \varepsilon}\left(\frac{B(a,b)}{\varepsilon A(a,b)}\right) > 0$$

whenever $B < 0$ and $A > 0$. Further, $\frac{d}{d\varepsilon} m_2 > 0$. It follows that for $0 < \varepsilon < \varepsilon_1$, the trajectory Δ_ε lies below Δ_{ε_1} as long as b is decreasing and a is increasing on Δ_{ε_1}. Suppose that $a^* < \bar{a} < a_2$ and the line $a = \bar{a}$ is to the left of where Δ_{ε_1} first intersects Γ_2, if this intersection occurs while $a' > 0$.

In the set where $a^* \leq a \leq \bar{a}$, $b \geq b_1$, and (a, b) is below Δ_{ε_1}, $\frac{B}{\varepsilon A} \to -\infty$ as $\varepsilon \to 0^+$, uniformly. Hence for sufficiently small ε the trajectory Δ_ε descends to below $b = b_1$ before $a = \bar{a}$. The rest of the lemma follows from the phase plane. □

For sufficiently small ε the trajectory Δ_ε just described blocks any possible periodic solution from crossing Γ_2 to the right of b^*, and so a phase plane argument shows that there are no small periodic solutions. The statement about limiting behavior in the theorem follows because for small ε either $\mathbf{p}(t)$ is close to Γ or else $\mathbf{p}'(t)$ is nearly vertical. On or near Γ_1, $A > 0$, while on or near Γ_3, $A < 0$.

For use below suppose that for each small $\eta > 0$,

$$\Lambda_\eta = \left\{(a,b) \in R \mid \min_{(x,y)\in\Gamma} \|(a,b) - (x,y)\| < \eta\right\},$$

where we use the euclidean norm on R^2. We require that

$$\eta < \min\{a_1 - c_1, c_2 - a_2, \frac{1}{\kappa}\}, \tag{2.4}$$

where κ is the maximum curvature of Γ.[3] This ensures that if the boundary of Λ_η is $\partial\Lambda_\eta$, then $\partial\Lambda_\eta$ consists of two smooth curves, one on either side of Γ. (See Figure 2.) The limiting argument above implies that for small ε (depending on η), a periodic trajectory can be considered to start in Λ_η with $a(0) = a^*$ and $b(0) < \beta^*$.

[3] Our hypotheses imply that κ is finite.

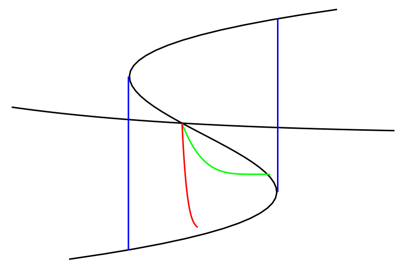

FIGURE 1. blackcurves:nullclines; red: γ_ϵ; green: γ_{ϵ_1}. The singular solution consists of the two vertical blue lines and the upper and lower curves which connect them. The red and green trajectories are described in the text.

2.2. **Proof of uniqueness of the periodic solution.** Using the same notation as that introduced just after (2.3), the spiralling behavior seen from phase plane analysis shows that for each (α^*, β) below Γ_1 there is a unique first $\tau = \tau(\beta) > 0$ such that

$$a(\tau, \beta) = a^* \text{ and } b(\tau, \beta) < \beta^*. \tag{2.5}$$

A solution is periodic only if

$$b(\tau(\beta), \beta) = \beta. \tag{2.6}$$

For small ε, any β for which (2.6) holds must be close to β^*. Let

$$\phi(\beta) = b(\tau(\beta), \beta) - \beta. \tag{2.7}$$

Uniqueness and asymptotic orbital stability follow by showing that there is an ε_0 such that if $0 < \varepsilon < \varepsilon_0$ and $\mathbf{p}_{\beta_1}$ is periodic, then $\phi'(\beta_1) < 0$.

The criterion for this is well-known. The periodic solution $(a(t, \beta_1), b(t, \beta_1))$ is asymptotically orbitally stable if

$$I(\beta_1) := \int_0^{\tau(\beta_1)} B_b(a(s, \beta_1), b(s, \beta_1))\, ds < 0. \tag{2.8}$$

(See for example [1].) If every periodic solution is asymptotically orbitally stable, then there is only one periodic solution.

We have the following key lemma:

Lemma 2. *Suppose that conditions C1-C5 are satisfied. Then there are positive numbers k_1, k_2, ν, and ε_2 such that for any ε with $0 < \varepsilon < \varepsilon_2$, and any β_1 in $(0, \beta^*)$ such that $\mathbf{p}_{\beta_1}$ is periodic,*

$$\frac{k_1}{\varepsilon} < \tau(\beta_1) < \frac{k_2}{\varepsilon}$$
$$I(\beta_1) < -\frac{\nu}{\varepsilon}.$$

Proof. The delicate part of the analysis is to consider the parts of the periodic trajectory which are close to the turning points (a_1, b_2) and (a_2, b_1). Indeed, these play an important role in all detailed studies of relaxation oscillations, such as [2] and [5]. In those papers, as in others, the equations are rescaled in these regions and either geometric perturbation theory or matched asymptotic analysis is used. In proving Theorem 1 we use no rescaling.

We will carry out this part of our analysis around the turning point (a_2, b_1), the other turning point being similar.

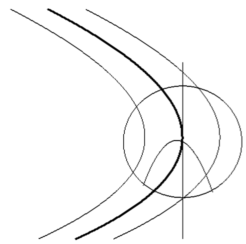

FIGURE 2. The region Λ_η between the two thin curves cuts through the disk D with $B_b < 0$ below the graph of f. S_η is the compact region between the vertical line and the right-most section of $\partial\Lambda_\eta$. $A(a,b) > 0$ in this region.

Recall the definitions of Γ and Λ_η in Section 2.1. Conditions C1-C5 imply that $B_b(a_2, b_1) = 0$, and that in some disk D centered at (a_2, b_1) the set of all points where $B_b = 0$ is a smooth curve $b = f(a)$ which intersects the boundary of D at exactly two points. (See Figure 2.)

The graph of f divides D into upper and lower parts. Because the tangent to Γ is vertical at (a_2, b_1) and $B_{bb}(a_2, b_1) > 0$, the subset of D where $B_b < 0$ is the lower part of D, and also we can choose the diameter of D small enough to ensure that $B_b < 0$ on $D \cap \Gamma \cap \{b < b_1\}$. (See Figure 2.) In addition we require that $\mathrm{diam}(D) < a_2 - a_*$. Note that D and the function f do not depend on η or ε. For small η, any trajectory in Λ_η below Γ_1 passes through D, and along such a trajectory $B_b < 0$ until after $a = a_2$.

For each small η, let S_η be the closure of the part of Λ_η to the right of $a = a_2$. There is an η_1 such that if $0 < \eta \leq \eta_1$ then S_η lies completely below the curve $A(a,b) = 0$, and so $A(a,b) > 0$ in the compact set S_{η_1}. We also require that η_1 satisfy (2.4). Let

$$A_1 = \min_{(a,b)\in S_{\eta_1}} A(a,b).$$

The result about limiting behavior implies that for each $\eta \in (0, \eta_1)$ we can choose an $\varepsilon_2(\eta)$ such that if $0 < \varepsilon < \varepsilon_2(\eta)$ then any periodic solution with $a(0) = a^*$, $b(0) = \beta_1 < \beta^*$ remains in Λ_η until after the point where it crosses $b = b_1$. Let the entire orbit of the solution be Q, and let Q_1 be the part of the orbit before it enters D. Let Q_2 be the part of the orbit between the boundary of D and $a = a_2$. Note that $B_b < 0$ on Q_2, so that

$$\int_{Q_2} B_b ds < 0.$$

As long after leaving Q_2 as $(a(t), b(t))$ is in S_η, $a'(t) \geq \varepsilon A_1$. As a result, the solution $\mathbf{p}_{\beta_1}$spends a time of at most $\frac{\eta}{\varepsilon A_1}$ in S_η, and when it exits S_η, to the right of $a = a_2$ and above $b = b_1$, it also exits Λ_η. For small ε the slope of $\mathbf{p}'_{\beta_1}$ is then large and positive until $\mathbf{p}_{\beta_1}$ crosses $A = 0$, and then it is large and negative until $\mathbf{p}_{\beta_1}$ is close to Γ_3 and the solution re-enters Λ_η.

Let Q_3 be the part of the orbit of $\mathbf{p}_{\beta_1}$ which is in S_η. If

$$M = \max_{(a,b)\in R} B_b(a,b),$$

then

$$\int_{Q_3} B_b ds \le M\frac{\eta}{\varepsilon A_1}.$$

Recall that B_b is negative on the relative interior of Γ_1. Hence, B_b is bounded above by a negative number on that part of Γ_1 outside of D. It follows that there is an $\eta_2 \in (0,\eta_1)$ such that $B_b < 0$ on

$$\Delta_{\eta_2} = \Lambda_{\eta_2} \cap D' \cap \{b \le b_1\}$$

where D' is the closure of the complement of the disk D. Recalling that D does not depend on η and $B_b < 0$ on $\Gamma_1 \backslash D$, we see that

$$-\lim_{\eta\to 0}\sup_{\Delta_\eta} B_b$$

is positive. Hence we can find an $\eta_3 \in (0,\eta_2)$ and a $\nu > 0$ such that if $0 < \eta < \eta_3$, $(a,b) \in \Delta_\eta$, and $A_2 = \max_R A$, then

$$-B_b(a,b)\frac{a_2 - a^* - \operatorname{diam}(D)}{A_2} > M\frac{\eta}{A_1} + \nu.$$

Suppose that $0 < \eta < \eta_3$. Then for sufficiently small ε any periodic solution spends a time of at least $\frac{a_2 - a^* - \operatorname{diam}(D)}{\varepsilon A_2}$ in Q_1, and

$$\int_{Q_1+Q_2+Q_3} B_b ds \le -\frac{\nu}{\varepsilon}.$$

A similar argument is applied along the entire top branch Γ_3, where also $\frac{dB}{\partial b} < 0$, and $A < 0$. Along the part of the solution in Λ_η near Γ_1 and to the left of a^* another large negative contribution is made to $I(\beta_1) = \int_Q B_b ds$. We may have to choose smaller ν, η_3, and $\varepsilon_2(\eta)$. Having fixed η, we saw earlier that within one period the solution spends a bounded time (independent of ε) outside of Λ_η, which implies that

$$I(\beta_1) < -2\frac{\nu}{\varepsilon} + H$$

for some constant H which is independent of ε and β_1.

The lower bound on $\tau(\beta_1)$ in the statement of the lemma has also been established. An upper bound follows easily since $|A|$ is bounded below by a positive number on $\Lambda_{\eta_1} \cap (\Gamma_1 \cup \Gamma_3)$. This completes the proofs of Lemma 2 and Theorem 1. □

3. The model of Kosiuk and Szmolyan

3.1. **Application of Theorem 1.** In [2] the functions εA and B contain additional parameters μ and δ, and are given by

$$\begin{aligned} \varepsilon A(a,b) &= \varepsilon\left(\mu\delta^2 - (1-\mu)a^2b^2\right) \\ B(a,b) &= a^2b^2\left(1+\delta^2-b\right) + \delta^2\left(\delta^2-b\right) \end{aligned} \tag{3.1}$$

The parameter μ lies in $(0,1)$. We will show that properties C1-C5 can then be verified for the A and B given above provided that $0 < \delta < \frac{1}{\sqrt{8}}$ and

$$\mu - 2\mu^2 - \delta^2 > 0. \tag{3.2}$$

In particular, this can only be true if $\mu < \frac{1}{2}$.

For $\delta \in \left(0, \frac{1}{\sqrt{8}}\right)$ let $c_1 = 0$, $c_2 = \sqrt{1-\delta^2}$, $d_1 = \delta^2$ and $d_2 = 1$. It follows that C1 is satisfied and that if

$$a^2 = \gamma\left(b\right)^2 = \frac{\delta^2\left(b-\delta^2\right)}{b^2\left(1+\delta^2-b\right)} \tag{3.3}$$

then $\Gamma \subset R$, $\gamma\left(d_1\right) = c_1$ and $\gamma\left(d_2\right) = c_2$, as in C2. We will check the rest of C2 below. The inequalities in C3 and C4 are either immediately obvious or follow because in this model $\frac{\partial B}{\partial b}$ is quadratic in b and negative when $b \le 0$. We need no further condition on δ to obtain a unique periodic solution for small ε.

In [2] there are results about the behavior of the solution as ε and $\delta \to 0$. In order to prove their entire main theorem, we must discuss this aspect of the problem, but first we complete our discussion of C1-C5. To check the remainder of C2 it must be shown that for $0 < \delta < \frac{1}{\sqrt{8}}$ the set of solutions to $B = 0$ is an S - shaped curve. We look for turning points by differentiating $B\left(a, b\right) = 0$ with respect to b implicitly and setting $\frac{da}{db} = 0$. This gives

$$a^2\left(2b\left(1+\delta^2\right) - 3b^2\right) = \delta^2. \tag{3.4}$$

Combining (3.3) and (3.4) gives

$$2b^2 - \left(1+4\delta^2\right)b + 2\delta^4 + 2\delta^2 = 0. \tag{3.5}$$

There will be two turning points if

$$\left(1+4\delta^2\right)^2 - 16\left(\delta^4+\delta^2\right) > 0,$$

which is true if $\delta^2 < \frac{1}{8}$, as mentioned before.

To check C5 it must be shown that there is a unique equilibrium point and it lies on the middle branch Γ_2 of Γ, where $\gamma' < 0$. (This will ensure that for small ε the equilibrium point is an unstable node or spiral.) The equilibrium point is found to be

$$\left(a^*, b^*\right) = \left(\frac{\sigma\delta}{\mu+\delta^2}, \mu+\delta^2\right), \tag{3.6}$$

where

$$\sigma = \sqrt{\frac{\mu}{1-\mu}} < 1. \tag{3.7}$$

The middle branch is characterized by having $B_b > 0$, and requiring that $B_b\left(a^*, b^*\right) > 0$ results in the inequality (3.2).

Finally we consider the case of small δ, and we find that the solutions of (3.5) are

$$b_1 = 2\delta^2 + O\left(\delta^4\right) \tag{3.8}$$

and

$$b_2 = \frac{1}{2} + O\left(\delta^2\right) \tag{3.9}$$

as $\delta \to 0$. From this and (3.3) it follows that

$$a_1 = 2\delta + O\left(\delta^3\right) \tag{3.10}$$

$$a_2 = \frac{1}{2} + \frac{1}{4}\delta^2 + O\left(\delta^4\right) \tag{3.11}$$

as $\delta \to 0$. In fact, from (3.3) we see that $\lim_{\delta\to 0}\gamma(b) = 0$ uniformly for b in any closed subinterval of $(0,1)$. Also, (3.3) implies that $b(1-b) \to 0$ as $\delta \to 0$ uniformly for a in any compact subinterval of $\left(0,\frac{1}{2}\right)$, and this, coupled with (3.8)-(3.11), implies that the singular solution of (3.1) tends as $\delta \to 0$ to the boundary C_0 of the rectangle $0 \le a \le \frac{1}{2}$, $0 \le b \le 1$.

3.2. **A more precise estimate.** The goal here is to complete the proof of Theorem 3.1 of [2] by showing that for the system (3.1), the ε_0 found in Theorem 1 can be chosen to depend linearly on δ, for small δ. For this it is convenient to follow [2] and set

$$\varepsilon = \tilde{\varepsilon}\delta \tag{3.12}$$

in (3.1). We therefore consider the system

$$\begin{aligned} a' &= \tilde{\varepsilon}\delta\left(\mu\delta^2 - (1-\mu)\, a^2b^2\right) \\ b' &= a^2b^2\left(1+\delta^2 - b\right) + \delta^2\left(\delta^2 - b\right). \end{aligned} \tag{3.13}$$

Theorem 2. *For each μ with $0 < \mu < \frac{1}{2}$ there are positive numbers $\tilde{\varepsilon}_0$ and δ_0 such that if $0 < \tilde{\varepsilon} < \tilde{\varepsilon}_0$ and $0 < \delta < \delta_0$, then the system (3.13) has a unique periodic solution and this solution is asymptotically orbitally stable. Further, $\tilde{\varepsilon}_0$ can be chosen so that for each $\eta > 0$ there is a δ_0 such that if $0 < \delta < \delta_0$ then the orbit P of the periodic solution satisfies*

$$dist(P, C_0) < \eta, \tag{3.14}$$

where

$$dist(P, C_0) = \min_{(p_1,p_2)\in P,(c_1,c_2)\in C_0} \max\left\{|p_1 - c_1|, |p_2 - c_2|\right\}.$$

Proof. (Figure 3 may be helpful in following the argument.) A straightforward computation shows that for any $\tilde{\varepsilon}_0$ there is a δ_0 such that if $0 < \tilde{\varepsilon} < \tilde{\varepsilon}_0$ and $0 < \delta < \delta_0$, then the equilibrium point of (3.13) is an unstable node. The existence of a periodic solution then follows from the standard phase plane argument, using the Poincaré-Bendixson theorem.

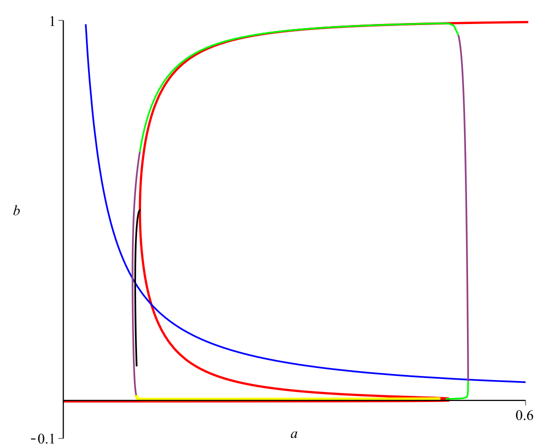

FIGURE 3. The red and blue curves are nullclines. Pieces $P_1, .., P_5$ from the lower right are green, purple, green, purple, yellow. The black curve on the left is described in the text.

Next we must modify Lemma 1. We do this in two parts.

Lemma 3. *There are $\tilde{\varepsilon}_0$ and δ_0 such that if $R_{\tilde{\varepsilon}_0,\delta_0}$ is the rectangle defined by $0 < \tilde{\varepsilon} < \tilde{\varepsilon}_0$ and $0 < \delta < \delta_0$, and $(\tilde{\varepsilon},\delta) \in R_{\tilde{\varepsilon}_0,\delta_0}$ then (a^*,b^*) is an unstable node, and the matrix*

$$M = \begin{pmatrix} \tilde{\varepsilon}\delta A_a & \tilde{\varepsilon}\delta A_b \\ B_a & B_b \end{pmatrix} |_{(a^*,b^*)}$$

has eigenvalues $\lambda_2 > \lambda_1 > 0$. Also, $\lambda_1 = O(\tilde{\varepsilon}\delta)$ and $\lambda_2 = B_b(a^*,b^*) + O(\tilde{\varepsilon}\delta)$ as $\tilde{\varepsilon}\delta \to 0$ within $R_{\tilde{\varepsilon}_0,\delta_0}$. An eigenvector $\mathbf{v} = (v_1,v_2)$ corresponding to λ_2 has slope $m_2 \le -\frac{c}{\tilde{\varepsilon}\delta}$ for some $c > 0$ independent of δ and $\tilde{\varepsilon}$ in $R_{\tilde{\varepsilon}_0,\delta_0}$.

Lemma 4. *Also, $(\tilde{\varepsilon}_0,\delta_0)$ can be chosen so that solutions (a,b) on the unique trajectory $\Delta_{\tilde{\varepsilon},\delta}$ tangent to $\mathbf{v}$ at (a^*,b^*) have $a' > 0$, $b' < 0$ as long as $b \ge \frac{5}{3}\delta^2$. Further, $b = \frac{5}{3}\delta^2$ before $a = -\delta\log\delta$. From $b = \frac{5}{3}\delta^2$, b continues to decrease, and a increase, until the trajectory crosses Γ_1. In particular, the orbit does not cross Γ_2.*

Remark 3. *Any number strictly between δ^2 and $2\delta^2$ can replace $\frac{5}{3}\delta^2$.*

From the phase plane and (3.8) it is clear that the trajectory considered in Lemma 4 crosses Γ_1 and then continues to the right and upward at least until $a = a_2$, at which point $b < b_1$. We prove these lemmas at the end of this section.

3.3. **Proof of limiting behavior assuming lemmas 3 and 4.** *Henceforth in this paper it will always be assumed that $(\tilde{\varepsilon},\delta) \in R_{\tilde{\varepsilon}_0,\delta_0}$. Thus, a restriction on $\tilde{\varepsilon}_0$ or δ_0 is automatically a restriction on $\tilde{\varepsilon}$ or δ.*

Since the orbit of a periodic solution must contain the equilibrium point (in its inside as a Jordan curve), Lemmas 3 and 4 imply that for sufficiently small $\tilde{\varepsilon}_0$ and δ_0 a closed orbit must also surround the turning point (a_2,b_1), because the orbit described in Lemma 4 blocks any possible smaller periodic solution. We can now assume that a periodic solution $\mathbf{p} = (a,b)$ starts with $a(0) = a_2$ and $0 < b(0) < b_1$. From (3.11) $a_2 > \frac{1}{2}$ if δ_0 is sufficiently small, and from (3.8) $b_1 = 2\delta^2 + O(\delta^4)$, as $\delta \to 0$. Recall that we denoted the orbit of $\mathbf{p}$ by P. We will divide this orbit into five pieces, which we denote by P_i, $i = 1..5$.

From the phase plane we see that starting at $(a(0), b(0))$, both a and b increase until the maximum of a, where $a' = 0$. Our first piece, P_1, is this segment of the orbit. We have the following bound as a start towards the limiting behavior result. Note that in this lemma δ_0 depends on $\tilde{\varepsilon}_0$.

Lemma 5. *For each $\eta > 0$, and any $\tilde{\varepsilon}_0$, δ_0 can be chosen so that if (a,b) is a periodic solution, then $a(t) \le \frac{1}{2} + \eta$ for all t.*

Proof. Let $\Lambda = \left\{(a,b) \mid a_2 \le a,\ 0 \le b \le \frac{1}{2}\right\}$. Then for small δ, $B \ge 0$ in Λ, by the definition of a_2. Also note that $\frac{\partial B}{\partial a} = 2ab^2\left(1+\delta^2-b\right) \ge \frac{1}{2}b^2$ in Λ, since $a_2 > \frac{1}{2}$. Hence $B \ge \frac{1}{2}b^2(a-a_2)$ in Λ. Also $a' \le \tilde{\varepsilon}\delta^3\mu$, and so while $a' > 0$ and $b \le \frac{1}{2}$,

$$\frac{db}{da} \ge \frac{b^2(a-a_2)}{2\tilde{\varepsilon}\mu\delta^3}.$$

Separating variables and integrating gives

$$b \ge b(0)\left(1 - \frac{b(0)(a-a_2)^2}{4\tilde{\varepsilon}\mu\delta^3}\right)^{-1} \ge \delta^2\left(1 - \frac{\delta^2(a-a_2)^2}{4\tilde{\varepsilon}_0\mu\delta^3}\right)^{-1} \tag{3.15}$$

as long as $a' > 0$ and $b \leq \frac{1}{2}$. But $a' = 0$ at $b = \frac{\sigma\delta}{a}$. Hence (3.8) and (3.15) imply that for any η we can choose δ_0 small, while keeping $\tilde{\varepsilon}_0$ fixed, to insure that $a' = 0$ before $a = \frac{1}{2} + \eta$. This proves Lemma 5. □

Thus for sufficiently small $\tilde{\varepsilon}_0$ and δ_0, the first piece of the orbit, P_1, ends at the unique point where $a' = 0$ and $a_2 < a < \frac{1}{2} + \eta$. From (3.13) it is seen that at this point, where P_2 begins, $b = a^{-1}\sigma\delta$.

From there b continues to increase, while a decreases. First consider (a, b) in the range $a \geq \frac{1}{4}$, say, and $a^{-1}\sigma\delta < b \leq \frac{1}{2}$. From (3.13) we find that in this region

$$\frac{db}{da} \leq -\frac{\left(a^2 b - \delta^2\right) b + O\left(\delta^4\right)}{\left(1 - \mu\right) \tilde{\varepsilon}_0 \delta a^2 b^2} \leq -\frac{c}{\delta}$$

for some c independent of $\tilde{\varepsilon}$ or δ. We conclude that for small δ_0, $b = \frac{1}{2}$ before $a = \frac{1}{2} - \sqrt{\delta_0}$.

In the region $\frac{1}{2} \leq b \leq 1 - \sqrt{\delta_0}$,

$$0 > a' \geq -\tilde{\varepsilon}_0 \delta_0 \left(1 - \mu\right) a^2$$

$$b' \geq \frac{1}{4} a^2 \sqrt{\delta_0} + O\left(\delta_0^2\right)$$

as $\delta_0 \to 0$. It follows that as long as $a \geq \frac{1}{4}$,

$$\frac{db}{da} \leq \frac{-a^2\sqrt{\delta} + O\left(\delta^2\right)}{4\tilde{\varepsilon}\delta\left(1 - \mu\right) a^2} \leq -\frac{1}{4\tilde{\varepsilon}_0\sqrt{\delta_0}} + O\left(\delta_0^2\right)$$

as $\delta_0 \to 0$, uniformly for $\tilde{\varepsilon} < \tilde{\varepsilon}_0$. Hence, for $\tilde{\varepsilon}_0 \leq \frac{1}{2}$ and small δ_0, $b = 1 - \sqrt{\delta_0}$ before $a = \frac{1}{2} - 2\sqrt{\delta_0}$. This is where P_2 ends. For small δ_0, $P_1 \cup P_2$ lies in

$$\left\{(a, b) \mid \frac{1}{2} - 2\sqrt{\delta_0} \leq a \leq \frac{1}{2} + \eta \text{ and } \delta^2 \leq b \leq 1 - \sqrt{\delta_0}\right\}. \tag{3.16}$$

From the end of P_2, a decreases at least until $a = a_1$. (See (3.10).) Meanwhile, b increases until P crosses Γ_3 above $b = 1 - \sqrt{\delta_0}$, and then $b' < 0$ at least until $b = b_1$. (See (3.9). Let P_3 denote the section of P from the end of P_2 until $b = \frac{2}{3}$ The relevance of $b = \frac{2}{3}$ will be seen later.

That the orbit has the behavior just described is seen from the phase plane (Figure 3). Since the solution is squeezed above Γ_3, it follows by the algebra of the previous section that as $(\tilde{\varepsilon}_0, \delta_0)$ tends to $(0, 0)$, P_3 approaches the union of the section of line $b = 1$ from $a = 0$ to $a = \frac{1}{2}$ and the section of the b - axis from $b = \frac{2}{3}$ to $b = 1$.

Piece P_4 of the orbit continues from $b = \frac{2}{3}$ down until $b = \frac{5}{3}\delta^2$, which occurs by Lemma 4 if $\tilde{\varepsilon}_0$ and δ_0 are sufficiently small. From there, where $b' < 0$, the trajectory P first crosses $a' = 0$ and then Γ_1, after which a and b increase and (a, b) returns to the starting point, where $a = a_2$. The final piece P_5 of P is that section from $b = \frac{5}{3}\delta^2$ to $a = a_2$.

Since $P_4 \cup P_5$ lies left of and below Γ_2, the algebraic argument of the previous section also shows that this union approaches the L - shaped path comprising the b axis from $b = \frac{2}{3}$ to $b = 0$ and the a - axis from $a = 0$ to $a = \frac{1}{2}$. This completes the proof of the limiting behavior of the solution as $\delta \to 0$, subject to the proofs of Lemmas 3 and 4.

3.4. **Proofs of Lemmas 3 and 4.**

Proof of Lemma 3. We use (3.6) and (3.13) to conclude that the linearized matrix at (a^*, b^*) is

$$M = \begin{pmatrix} -2\tilde{\varepsilon}\delta^2\sqrt{\mu(1-\mu)}\left(\mu+\delta^2\right) & -\frac{2\tilde{\varepsilon}\mu\delta^3}{\mu+\delta^2} \\ 2\sqrt{\mu(1-\mu)}\left(\mu+\delta^2\right)\delta & \frac{1-2\mu}{1-\mu}\delta^2 + O\left(\delta^4\right) \end{pmatrix}$$

as $\delta \to 0$, uniformly for $\tilde{\varepsilon} < \tilde{\varepsilon}_0$. Hence, as $(\tilde{\varepsilon}, \delta) \to (0,0)$, and recalling (3.7),

$$\lambda_1 + \lambda_2 = \left(\frac{1-2\mu}{1-\mu} + O(\tilde{\varepsilon})\right)\delta^2 + O\left(\delta^4\right) = B_b(a^*, b^*) + O\left(\tilde{\varepsilon}\delta^2\right) + O\left(\delta^4\right)$$
$$\lambda_1\lambda_2 = \tilde{\varepsilon}\mu\left(-2\sigma(1-2\mu) + 4\sqrt{\mu(1-\mu)}\right)\delta^4 + O\left(\tilde{\varepsilon}\delta^6\right),$$

again uniformly in $R_{\tilde{\varepsilon}_0,\delta_0}$. Since $0 < \mu < \frac{1}{2}$ in our theorem, $\tilde{\varepsilon}_0$ and δ_0 can be chosen so that the conclusions of the first two sentences of the lemma hold. Also, an eigenvector $\mathbf{v}$ corresponding to λ_2 satisfies

$$v_2 = \frac{B_a(a^*, b^*)}{\lambda_2 - B_b(a^*, b^*)} v_1.$$

Because $A_a(a^*, b^*) < 0$, $B_b(a^*, b^*) > 0$, and $\tilde{\varepsilon}\delta A_a + B_b = \lambda_1 + \lambda_2$, it follows that $\lambda_2 < B_b$ and so the slope of $\mathbf{v}$ is negative and satisfies the final conclusion of the lemma. □

Proof of Lemma 4.

Lemma 6. *Suppose that* $0 < \hat{b} < \mu$. *Let* $r = \frac{1}{2}(1+\sigma)$. *Then* $(\tilde{\varepsilon}_0, \delta_0)$ *can be chosen so that the trajectory* $\Delta_{\tilde{\varepsilon},\delta}$ *stays to the left of* $a = \frac{r}{b^*}a^*$ *as long as* $b \geq \hat{b}$.

Proof. For this proof it is convenient to use a scaling similar to ones used in [7] and [2], letting $\tau = \delta^2 t$ and

$$\begin{aligned} a(t) &= \delta P(\tau) \\ b(t) &= Q(\tau). \end{aligned} \tag{3.17}$$

Then

$$\begin{aligned} P' &= \tilde{\varepsilon}\left(\mu - (1-\mu)P^2Q^2\right) \\ Q' &= P^2Q^2\left(1+\delta^2 - Q\right) + \delta^2 - Q. \end{aligned} \tag{3.18}$$

From (3.6) it is seen that the equilibrium point for this system is

$$(p^*, q^*) = \left(\frac{\sigma}{\mu+\delta^2}, \mu+\delta^2\right).$$

Note that $q^* = b^*$. We consider (3.18) in the compact region $1 \leq P \leq \frac{r}{q^*}p^*$, $\hat{b} \leq Q \leq \frac{2}{3}$. In that region, this system is a regular perturbation from the case $\delta = 0$. For $\delta = 0$, $q^* = \mu$, and the argument used in the proof of Lemma 1 shows that on the orbit Δ_ε as defined in Lemma 1, for sufficiently small $\tilde{\varepsilon}_0$, Q decreases to $Q = \hat{b}$ before $P = \frac{r}{q^*}p^*$. The same is then true for sufficiently small δ_0 (with possibly a smaller $\tilde{\varepsilon}_0$), and changing back to the (a, b) coordinates completes the proof of Lemma 6. □

Note that in contrast to the proof of Lemma 1, this argument is not valid down to b_1, because when $\delta = 0$ in (3.18) there is no second turning point. Hence, to complete the proof of Lemma 4 we must continue the solution from $b = \hat{b}$ (to be chosen later) to $b = \frac{5}{3}\delta^2$ (where obviously we must assume that $\frac{5}{3}\delta^2 < \hat{b}$).

We will show that for sufficiently small $\tilde{\varepsilon}_0$ and δ_0, $b = \frac{5}{3}\delta^2$ before $a = -\delta \log \delta$. Since $b_1 = 2\delta^2 + O\left(\delta^4\right)$ as $\delta \to 0$, a simple phase plane analysis then shows that the solution must cross the lower branch Γ_1 of the b - nullcline.

As long as $a' > 0$ we can write

$$\frac{db}{da} = \frac{\left(a^2 b\left(1+\delta^2\right) - a^2 b^2 - \delta^2\right) b + \delta^4}{\tilde{\varepsilon}\delta\left(\mu\delta^2 - (1-\mu)\, a^2 b^2\right).} \tag{3.19}$$

If $a' > 0$ then the denominator of (3.19) is less than $\tilde{\varepsilon}\mu\delta^3$. For the numerator we need to estimate $a^2 b$ from the point where $b = \hat{b}$. Using (3.13) we find that

$$\frac{d}{dt}\left(a^2 b\right) \leq 2\tilde{\varepsilon}\delta^3\mu a b + a^2\delta^4 + \left(-\delta^2 + a^2 b\left(1+\delta^2\right)\right) a^2 b. \tag{3.20}$$

By Lemma 6 and (3.6), if $a = \hat{a}$ when $b = \hat{b}$ then $a^* < \hat{a} \leq \kappa\delta$, where $\kappa = \frac{r}{\mu^2}\sigma$.

Let $\hat{b} = \frac{1}{6\kappa^2}$. Then $\hat{a}^2\hat{b}\left(1+\delta^2\right) \leq \frac{1}{3}\delta^2$ (for $\delta < 1$). If $\frac{5}{3}\delta^2 \leq b \leq \hat{b}$, then

$$a^2\delta^4 \leq \frac{3}{5}\delta^2 a^2 b. \tag{3.21}$$

Also it is easily seen that for small δ_0, $a^* \geq 2\delta$. If $a \geq 2\delta$ then

$$2\tilde{\varepsilon}\delta^3\mu a b \leq \tilde{\varepsilon}\mu\delta^2 a^2 b.$$

Hence, as b decreases from $\hat{b}$, if $a^2 b\left(1+\delta^2\right) \leq \frac{1}{3}\delta^2$ then

$$\left(a^2 b\right)' \leq \left(-1 + \frac{1}{3} + \frac{3}{5} + \tilde{\varepsilon}\mu\right)\delta^2\left(a^2 b\right) < 0.$$

Thus, $a^2 b\left(1+\delta^2\right) \leq \frac{1}{3}\delta^2$ as long as $b \geq \frac{5}{3}\delta^2$ and $\tilde{\varepsilon}_0\mu < \frac{1}{15}$.

To complete the proof of Lemma 4 we will show that b reaches $\frac{5}{3}\delta^2$ before $a = -\delta \log \delta$. We now see that

$$\begin{aligned} b' &\leq \left(a^2 b\left(1+\delta^2\right) - \delta^2\right) b + \delta^4 \\ &\leq -\frac{2}{3}\delta^2 b + \delta^4 \leq \left(-\frac{2}{3} + \frac{3}{5}\right)\delta^2 b \end{aligned}$$

as long as $b \geq \frac{5}{3}\delta^2$.

As we saw earlier, if $a' > 0$ then $a' < \tilde{\varepsilon}\mu\delta^3$, and so

$$\frac{db}{da} \leq -\frac{1}{15\tilde{\varepsilon}\mu\delta} b.$$

Integrating this inequality from $\hat{a}$ to $-\delta \log \delta$ and using the choice of $\hat{b}$ given above shows that we need

$$\frac{1}{6\kappa^2}\left(e^{\kappa}\delta\right)^{\frac{1}{15\tilde{\varepsilon}\mu}} \leq \frac{5}{3}\delta^2.$$

This will be true for small δ_0 if $15\tilde{\varepsilon}_0\mu < \frac{1}{2}$, completing the proof of Lemma 4. $\square$

3.5. **Uniqueness and stability.** As discussed in Section 2.2, to complete the proof of Theorem 2 we must show that $\tilde{\varepsilon}_0$ and δ_0 can also be chosen so that if (a,b) is a periodic solution, say with period T, then

$$\int_0^T B_b\left(a\left(t\right),b\left(t\right)\right)dt<0.$$

Here

$$B_b\left(a,b\right)=2a^2b\left(1+\delta^2\right)-3a^2b^2-\delta^2 \tag{3.22}$$

As before, it is always assumed that $(\tilde{\varepsilon},\delta)\in R_{\tilde{\varepsilon}_0,\delta_0}$. We begin now with piece P_3, starting at $b=1-\sqrt{\delta_0}$, $a\geq\frac{1}{2}-2\sqrt{\delta_0}$. The situation is in some ways easier than that considered in Section 2.2. Observe that $B_b\left(a,b\right)<0$ if $b\geq\frac{2}{3}$. Also, (3.3) implies that if $a>2\sqrt{\delta}$ and $b\geq\frac{1}{2}$ then $b\geq1-\sqrt{\delta_0}$ for all sufficiently small δ.

It further is seen from (3.22) that if $b\geq1-\sqrt{\delta_0}$ then

$$B_b\left(a,b\right)=-a^2+O\left(\sqrt{\delta_0}\right) \tag{3.23}$$

as $\delta\to0$, uniformly in any compact subinterval of $0<a<\frac{1}{2}$. Also, our bounds on μ, a, and b show that on $P_3\cap\left\{b\geq1-\sqrt{\delta_0}\right\}$,

$$0>a'>-\tilde{\varepsilon}_0\delta_0. \tag{3.24}$$

Further, B_b is negative on all of P_3.

Since $a'<0$ on P_3, and so the inverse $t\left(a\right)$ of a exists,

$$\int_{P_3}B_bdt:=\int_{t_0}^{t_1}B_b\left(a\left(t\right),b\left(t\right)\right)dt=\int_{P_3}\frac{B_b}{a'}da:=\int_{a(t_0)}^{a(t_1)}\frac{B_b\left(a,b\left(t\left(a\right)\right)\right)}{a'\left(t\left(a\right)\right)}da$$

where $a\left(t_0\right)$ and $a\left(t_1\right)$ are the values of a at the right and left endpoints of P_3. It follows from (3.23) that

$$\lim_{(\tilde{\varepsilon}_0,\delta_0)\to(0,0)}\tilde{\varepsilon}\delta\int_{P_3}B_bdt\leq\int_{\frac{1}{2}}^{0}a^2da=-\frac{1}{24}. \tag{3.25}$$

The piece P_4 descends from $b=\frac{2}{3}$ to $b=\frac{5}{3}\delta^2$. We wish to show that it does not come too close to the equilibrium point (a^*,b^*). For this purpose we again use (3.17). We consider the solution with $\left(P\left(0\right),Q\left(0\right)\right)=\left(\frac{a_1}{\delta},b_2\right)$, which is the left upper turning point of the nullcline $Q'=0$. At this point, $Q'=0,P'<0$, but writing

$$\frac{dQ}{dP}=\frac{P^2Q^2\left(1+\delta^2-Q\right)+\delta^2-Q}{\tilde{\varepsilon}\left(\mu-\left(1-\mu\right)P^2Q^2\right)}$$

except at $P'=0$, it is easily shown that for small δ_0 and $\tilde{\varepsilon}_0$ the solution turns down and rapidly descends, say to $Q=\hat{b}$, the number chosen during the proof of Lemma 4. For sufficiently small $\tilde{\varepsilon}_0$ we can conclude that $a=\delta P\leq\frac{1}{2}\left(a_1+a^*\right)$ as long as $b\geq\hat{b}$. This solution, which is shown in black on Figure 3 starting at about $(a,b)=(.1,.5)$, blocks the periodic orbit from coming too close to (a^*,b^*). To say this carefully, there is a $c>0$ (perhaps different from previous c's we have used) such that on the piece P_4, for sufficiently small δ

$$\left(\frac{a-a^*}{\delta}\right)^2+\left(b-b^*\right)^2\geq c$$

as long as $(a,b) \in P_4$ and $b \geq \hat{b}$. This follows because for $Q \geq \hat{b}$ the system for (P,Q) is a regular perturbation from the case $\delta = 0$, and $P = \frac{a}{\delta}$, $Q = b$.

We then conclude, for some further positive c's independent of δ and ε, that with $\tau = \delta^2 t$, $\left(\frac{dP}{d\tau}\right)^2 + \left(\frac{dQ}{d\tau}\right)^2 \geq c$. Hence (P,Q) is in P_4 for a τ - interval of length $\Delta\tau_{P_4} \leq c$. In the original variable, $\Delta t_{P_4} \leq \frac{c}{\delta^2}$. On P_4, $a \leq \delta\left|\log\delta\right|$ and $b \leq \frac{2}{3}$, so from (3.22), $B_b\left(a.b\right) = O\left(\delta^2\log^2\delta\right)$ and

$$\int_{P_4} B_b dt = O\left(\log^2\delta\right) \tag{3.26}$$

as $\delta \to 0$, uniformly for $0 < \tilde{\varepsilon} < \tilde{\varepsilon}_0$. While this is not small, it is dominated by the negative estimate obtained from (3.25).

On P_5, $B_b < 0$. One way to see this is to show that $B_{bb}(a_2, b) > 0$ for $b \leq b_1$ (using (3.8)), and so $B_b\left(a_2, b\right) < 0$ for $0 < b < b_1$. Then show that $B_{ba} > 0$ for $b < b_1$, implying that $B_b < 0$ to the left of the segment $a = a_2$, $0 < b < b_1$. Hence,

$$\int_{P_5} B_b dt < 0. \tag{3.27}$$

Finally, consider $\hat{P} = P_1 \cup P_2$. We only have to look at the region where possibly $B_b > 0$, which is to say $b \leq \frac{2}{3}$, and we write this as $\left(\hat{P} \cap \left\{b \leq 25\delta^2\right\}\right) \cup \left(\hat{P} \cap \left\{25\delta^2 < b \leq \frac{2}{3}\right\}\right)$. If $b \leq 25\delta^2$ then $B_b \leq 50a^2\left(1+\delta^2\right)\delta^2$. Also, $a' \geq \tilde{\varepsilon}\delta\left(\mu\delta^2 - 25^2\delta^4\right)$, since $a^2 \leq 1$. We can then chose δ_0 so small that $a' \geq \frac{1}{2}\tilde{\varepsilon}\mu\delta^3$ in the interval under discussion. In fact, we can take $a^2\left(1+\delta^2\right) < 1$, and so by the bounds just above,

$$\int_{\hat{P}\cap\{b\leq 25\delta^2\}} B_b dt \leq \int_{a_1}^{\frac{1}{2}+\eta} \frac{2B_b\left(a,b\right)}{\tilde{\varepsilon}\mu\delta^3} da \leq \frac{100\eta}{\tilde{\varepsilon}\mu\delta}.$$

We choose η so that $100\eta < \frac{\mu}{26}$, and obtain that

$$\int_{\hat{P}\cap\{b\leq 25\delta^2\}} B_b dt \leq \frac{1}{26\tilde{\varepsilon}\delta}.$$

Recall from (3.16) that $a \geq \frac{1}{2} - 2\sqrt{\delta_0}$ on $\hat{P}$, and so if $b > 25\delta^2$ then for sufficiently small δ_0, $\delta^2 b < \frac{1}{6}a^2b^2$ on $\hat{P}$. If in addition $b \leq \frac{2}{3}$ then

$$B \geq a^2b^2\left(1 + \delta - \frac{2}{3}\right) - \frac{1}{6}a^2b^2,$$

and so for sufficiently small $\tilde{\varepsilon}_0$ and δ_0,

$$\frac{B_b}{B} \leq \frac{12a^2b\left(1+\delta^2\right)}{a^2b^2} \leq \frac{13}{b}$$

on $\hat{P} \cap \left\{25\delta^2 < b \leq \frac{2}{3}\right\}$. Hence

$$\int_{\hat{P}\cap\{b>25\delta^2\}} B_b dt \leq \int_{25\delta^2}^{\frac{2}{3}} \frac{13}{b} db = O\left(-\log\delta\right)$$

as $\delta \to 0$. The constant in this bound is independent of $\tilde{\varepsilon}_0$, which implies that for sufficiently small δ_0,

$$\int_{P_1\cup P_2} B_b dt \leq \frac{1}{25\tilde{\varepsilon}\delta}. \tag{3.28}$$

Combining (3.25), (3.26), (3.27), and (3.28) completes the proof of Theorem 2.

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