On relaxation times in the Navier-Stokes-Voigt model

William J. Layton * Leo G. Rebholz †

Abstract

We give two results that indicate that the relaxation time for the flow governed by the Navier-Stokes-Voigt (NSV) model is sinificantly larger than for the Navier-Stokes equations. We first show that for the Green-Taylor vortex decay problem, NSV admits an exact solution which has a significantly larger half life than for true fluid flow. Second, we observe in a channel flow test that NSV provides more regular solutions than usual Navier-Stokes solutions but NSV approximations take significantly longer to reach the steady state.

1 Introduction

An evolution equation $u_t + N(u) = f$ can be smoothed by the pseudoparabolic regularization

$$u_t^\alpha + \alpha^2 A u_t^\alpha + N(u^\alpha) = f,$$

where $\alpha > 0$ is small and A is a positive operator. This regularization occurs naturally in the modified equations of many splitting methods, whose literature extends to the late 1950's. An early systematic study of it was in [5].

As a numerical tool, it can arise by adding numerical diffusion at one time level and antidiffusing at the previous level, and its use has seen success when used with Navier-Stokes equations (NSE) [10], ocean models [1], MHD [9], and other related systems. This pseudoparabolic regularization of the NSE was proposed by Voigt in 1892 for certain viscoelastic fluids [15] and analyzed by Oskolkov [12]. The resulting Navier-Stokes-Voigt (NSV) model takes the form

$$v_t - \alpha^2 \Delta v_t + v \cdot \nabla v + \nabla q - \nu \Delta v = f, \tag{1.1}$$

$$\nabla \cdot v = 0. \tag{1.2}$$

Here v and q are the NSV velocity and pressure, f the forcing, ν the kinematics viscosity, and $\alpha > 0$ is a regularization parameter with units of length. The usual NSE are recovered when $\alpha = 0$.

The NSV model has the same steady state solutions as the NSE, is globally well-posed under homogeneous Dirichlet boundary conditions, and thus does not need additional or ad-hoc boundary conditions, [3]. Finite element methods analysis of NSV in [10, 9] shows that when temporally discretized with Crank-Nicolson, i) NSV implementation is a simple change from a Crank-Nicolson discretization of usual NSE, as it only involves changing coefficients, ii) it acts as a stabilization method that improves conditioning for higher Reynolds number problems, and iii) the error (to the

*Department of Mathematics, University of Pittsburgh, wjl@pitt.edu, partially supported by NSF grant 1216465

†Department of Mathematical Sciences, Clemson University, rebholz@clemson.edu, partially supported by NSF grant DMS1112593.

NSE solution) introduced by NSV is no worse asymptotically than that of the NSE discretization itself if Taylor-Hood mixed finite elements are used and α is chosen of the order of the mesh width.

These attractive properties led to preliminary tests in [8] of NSV as a tool for 'spin up'. The question was: Does the NSV ($\alpha > 0$) solution approach equilibrium or statistical equilibrium faster than the Navier-Stokes ($\alpha = 0$) solution? Surprisingly, in [8], the opposite was reported. Our aim herein is to explain through simple examples, including the Green-Taylor vortex and a simple channel flow experiment, why this was observed.

2 Taylor-Green vortex

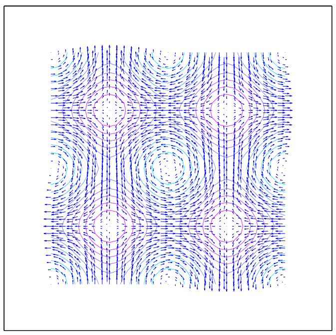

Figure 1: Velocity field and pressure contours of the Taylor-Green vortex, with n=2.

We consider first the Green-Taylor vortex decay problem [6, 14], which is an exact solution of the NSE with no forcing and periodic boundary conditions. In $\Omega = (0,1) \times (0,1)$, solutions take the form

$$\begin{aligned} u_1(x,y,t) &= -\cos(n\pi x)\sin(n\pi y)e^{-2n^2\pi^2\nu t} \\ u_2(x,y,t) &= \sin(n\pi x)\cos(n\pi y)e^{-2n^2\pi^2\nu t} \\ p(x,y,t) &= -\frac{1}{4}(\cos(2n\pi x)+\cos(2n\pi y))e^{-4n^2\pi^2\nu t} \end{aligned}$$

where n can be chosen as any positive integer. This exact NSE solution is made of an $n \times n$ array of oppositely signed vortices that decay as $t \to \infty$. Figure 1 shows the velocity field and pressure contours for the test problem, with $n = 2$. It has been used as a numerical test in [4], [13], and [7], and many other papers. The Green-Taylor vortex solution above is easily extended to an exact solution of the NSV given by

$$\begin{aligned} v_1(x,y,t) &= -\cos(n\pi x)\sin(n\pi y)e^{\frac{-2n^2\pi^2\nu t}{1+2n^2\pi^2\alpha^2}} \\ v_2(x,y,t) &= \sin(n\pi x)\cos(n\pi y)e^{\frac{-2n^2\pi^2\nu t}{1+2n^2\pi^2\alpha^2}} \\ q(x,y,t) &= -\frac{1}{4}(\cos(2n\pi x)+\cos(2n\pi y))e^{\frac{-4n^2\pi^2\nu t}{1+2n^2\pi^2\alpha^2}}. \end{aligned}$$

Note that the NSE ($\alpha = 0$) solution and the NSV ($\alpha > 0$) solution have the same spacial patterns, but differ in time. The NSV solutions decay at a slower rate. We calculate the half life of the respective solutions to be, respectively,

$$T_{1/2}^{NSE} = \frac{\log(2)}{4n^2\pi^2\nu}\ , \quad T_{1/2}^{NSV} = \frac{\log(2)}{4n^2\pi^2\nu}(1+2n^2\pi^2\alpha^2),$$

and so

$$T_{1/2}^{NSV} = (1+2n^2\pi^2\alpha^2)T_{1/2}^{NSE} > T_{1/2}^{NSE}.$$

Thus, the NSV solution can take significantly longer to reach equilibrium than the NSE solution. Further, the larger α becomes and the smaller the spacial scales present (as n increases) in the solution, the longer the relaxation time required.

3 A channel flow test

We test here NSV on a channel flow with a cold start. Due to the cold start, the nonlinearity is active until the flow reaches a Poiseuille steady state (for which the nonlinearity is zero). We compare approximations of NSV and NSE found by a common discretization method: Crank-Nicolson in time, (P_2, P_1) Taylor-Hood finite elements in space (see, e.g. [11]). In addition to being a very common discretization method, it is quite attractive for use with NSV because changing an NSE code to NSV involves simply changing two coefficients, as seen below in (3.1). For a regular, conforming triangulation τ_h of a domain Ω, defining $(X_h, Q_h) := (P_2(\tau_h)^2, P_1(\tau_h))$ to be the usual, LBB stable Taylor-Hood pair to approximate velocity and pressure spaces. The timestepping scheme we use is as follows: Denoting the $L^2(\Omega)$ inner product by $(\cdot,\cdot)$ and $\phi^{n+1/2} := 1/2(\phi^n+\phi^{n+1})$, for a given timestep $\Delta t > 0$ and regularization parameter α (choose $\alpha = 0$ for NSE, $\alpha > 0$ for NSV), and $v_h^n \in X_h$ satisfying $(\nabla \cdot v_h^n, r_h) = 0\ \forall r_h \in Q_h$, find $v_h^{n+1},\ q_h^{n+1} \in X_h \times Q_h$ satisfying $\forall \chi_h,\ r_h \in X_h \times Q_h$,

$$\begin{aligned} \frac{1}{\Delta t}\left(v_h^{n+1} - v_h^n, \chi_h\right) + \left(v_h^{n+1/2}\cdot\nabla v_h^{n+1/2}, \chi_h\right) - \left(q_h^{n+1}, \nabla\cdot\chi_h\right) & \\ + \left(\frac{\nu}{2} + \frac{\alpha^2}{\Delta t}\right)\left(\nabla v_h^{n+1}, \nabla\chi_h\right) + \left(\frac{\nu}{2} - \frac{\alpha^2}{\Delta t}\right)\left(\nabla v_h^n, \nabla\chi_h\right) &= \left(f(t^{n+1/2}), \chi_h\right), \quad (3.1) \\ \left(\nabla\cdot v_h^{n+1}, r_h\right) &= 0. \quad (3.2) \end{aligned}$$

For more specifics on this well-known and widely used scheme for the NSE, see [11], and for NSV see [9, 10].

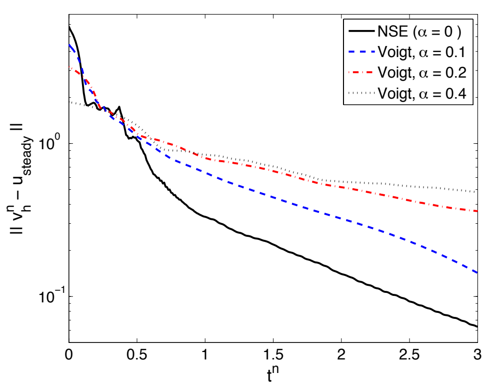

Figure 2: Convergence to the steady solution for NSE ($\alpha = 0$) and NSV with varying α. As α is increases, convergence is slowed in time.

The domain for this test problem is $\Omega = (0,4)\times(0,1)$, which represents a channel. We enforce a parabolic inflow

$$u(0,y,t) = \begin{pmatrix} \frac{1}{2\nu} y(1-y) \\ 0 \end{pmatrix}, \quad t > 0,$$

no slip boundary conditions on the walls, a zero-traction outflow enforced weakly with the 'do-nothing' condition, and start the flow from rest ($u_0 = \langle 0, 0\rangle^T$). For our tests, we choose $\nu = 0.002$, which is sufficiently small so that the NSE flow development is not initially smooth.

We compute using (3.1)-(3.2) with a Delaunay mesh with 4,104 degrees of freedom, $\Delta t = 0.01$ and endtime $T = 3$. It has been observed [2] that the true velocity reaches a steady state given by

$$u(x,y,t) = \begin{pmatrix} \frac{1}{2\nu} y(1-y) \\ 0 \end{pmatrix}.$$

We run the computations with α=0, 0.1, 0.2, and 0.4, and compare solutions in the plots below. First, we plot the difference between the approximated solutions and the steady (Poiseuille) solution, versus time, in figure 3. Here we observe that all solutions appear to be tending towards the correct steady solution, but for larger α convergence is much slower. Figure 3 shows velocity streamlines over speed contours for the NSE and NSV (with $\alpha = 0.4$), respectively, at t=0.1, 0.2, 0.5 and 1.0. Here we observe that NSV provides smoother solutions, particularly at earlier times when the NSE is still developing, but it takes longer for the higher speed from the inlet to progress through the channel.

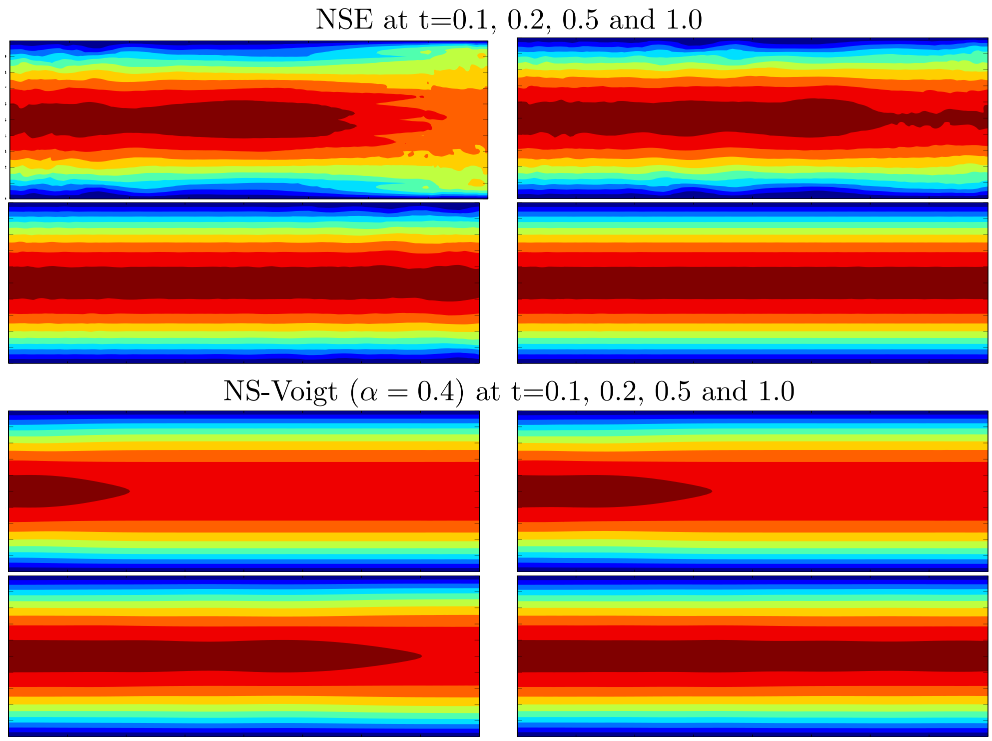

Figure 3: Speed contours for the NSE and NSV ($\alpha = 0.4$) solutions at t=0.1, 0.2, 0.5, and 1.0. For NSV, it takes longer for the high speed, central core flow to reach the end of the channel.

In Figure 3 we illustrate the mode of convergence by giving the evolution of the speed contours. Note the difference between that the fastest, central part of the two flows. The NSE solution is rougher but the fast, central flow moves rapidly across the channel. The NSV solution is smoother but develops significantly more slowly.

4 Conclusions

Knowing a model's drawbacks is as important as knowing a model's positive features We have found that flow evolution governed by NSV can have significant temporal error compared to true fluid flow due to longer relaxation time. Specifically, NSV gives an accurate approximation of long time averaged statistical properties of a flow but its predicted evolution to statistical equilibrium is significantly modified as described above.

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