OPTIMAL CONTROL OF SYSTEMS GOVERNED BY PDES WITH RANDOM PARAMETER FIELDS

ROXANA TANASE*, CATALIN TRENCHEA †, AND IVAN YOTOV ‡

Abstract. We present methods for the optimal control / parameter identification of systems governed by partial differential equations with random input data. We consider several identification objectives that either minimize the expectation of a tracking cost functional or minimize the difference of desired statistical quantities in the appropriate spatial-L^p norm (including higher order moments, hence allowing to match any statistics, e.g., the variance, skewness, kurtosis, etc.).

A specific problem of parameter identification of a linear elliptic PDE that describes flow of a fluid in a porous medium with uncertain permeability field is examined. We present numerical results to study the consequences of the moment-tracking approximation and the efficiency of the method. The stochastic parameter identification algorithm integrates an adjoint-based deterministic algorithm with the sparse grid stochastic collocation mixed-FEM approach.

We also derive rigorous error estimates for fully discrete problems, using the Fink-Rheinboldt theory for the approximation of solutions of a class of nonlinear problems.

Key words.

AMS subject classifications.

1. Introduction.

Driven by the needs from applications both in industry and other sciences, the field of inverse problems has undergone a tremendous growth within the last two decades, where recent emphasis has been laid more than before on nonlinear problems. Advances in this field and the development of sophisticated numerical techniques for treating the direct problems allow to address and solve industrial inverse problems on a level of high complexity.

Parameter estimation is an important field in the area of modeling physical or biological processes. The set of parameters that maximize the model's agreement with experimental data, i.e. the ideal parameter set, can be used to yield important insight into a given system. It can help scientists more clearly describe the behavior of the system, predict behavioral changes in the system during pathological situations, and assess the efficacy of various corrective options. In addition, once those ideal parameters have been found, other mathematical techniques can be used to obtain further insight into the system's behavior. Local sensitivity analysis at the optimal parameter set can be used to assess the local importance of the parameters.

As mathematical/computational models become more complex in order to better describe physical systems, parameter estimation can grow in difficulty and cost due to the increase in number of parameters and consequently computational runtime. The problem of calibrating a model in a reasonable amount of time depends more and more on efficient methods of parameter estimation.

There is a vast literature on estimating parameters that arise in partial differential equations using different techniques. The most studied approach to stochastic inverse

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problems is the Bayesian approach [57, 42, 56, 35, 14, 10, 37, 46, 45, 47, 44, 6, 23, 51, 17, 24, 39, 3, 43, 2, 11].

Following [28], we consider the other approach - the adjoint-variable method. It relies on deterministic simulators, and unlike the Bayesian approach, requires no a priori assumptions on the design variables. The first type of cost functional we investigate with this approach, which is in ubiquitous use in the literature (see e.g., [5, 13, 12, 34, 33, 59]), is defined as the expected value of a cost functional used in deterministic control and identification problems. The second type, we apply the deterministic cost functional to the expected value of the mismatch, we attempt to minimize the spatial L^2 mismatch of the expected values (and/or higher-order moments) of the solutions of the PDE and the given target function. This type of functional was considered from a computational point of view and for a small number of parameters in [60].

The adjoint variable-based method solves a large class of optimization, inverse problems, parameter estimation and optimal control problems. It is one of the gradient-based techniques in which gradient vector of the cost functional with respect to the unknown parameters is calculated indirectly by solving an adjoint equation. Although an additional cost arises from solving the adjoint equation, the gradients of the cost functional can be altogether achieved with respect to each parameter. Thus, the total cost to obtain these gradients is independent of the number of parameters and amounts to the cost of solving two partial differential equations (PDEs) roughly. From a control theory point of view, the algorithm is based on the Pontryagin maximum principle, since it tries to iteratively solve the necessary conditions for optimality. From an optimization point of view, the algorithm consists of a gradient descent, in which the gradient of the cost functional is efficiently computed via the adjoint variable-based method.

The paper is organized as follows. In §2, we state the specific PDE we consider as a constraint for the stochastic inverse problems, and also list assumptions about that PDE and the nature of the stochastic inputs we consider. In §3, we precisely state the stochastic control and identification problems we consider, including the definitions of the two types of cost functionals. Existence and uniqueness results are stated and the first-order optimality conditions which optimal states and controls or parameters, as the case may be, must satisfy are derived. Finally, in §5, we provide some numerical examples which we use to illustrate the better matching results as well as the better computational efficiency resulting from using the second approach for defining stochastic cost functionals.

2. Problem setting. Let D be a convex bounded polygonal domain in $\mathbb{R}^d$, $d = 1, 2, 3$, and $(\Omega, \mathcal{F}, P)$ a complete probability space, where Ω is the set of outcomes, $\mathcal{F} \subset 2^\Omega$ is the σ-algebra of events and $P : \mathcal{F} \to [0, 1]$ is a probability measure. The general framework for the stochastic inverse problem is the following: we seek random parameters, coefficients $\kappa(\omega, x)$ and/or forcing terms $f(\omega, x)$, with $x \in D$, $\omega \in \Omega$, that minimize the mismatch between stochastic measured and simulated data. We denote by $W(D)$ a Banach space of functions $v : D \to \mathbb{R}$ and define the stochastic Banach space $L^2_P(\Omega; W(D))$, consisting of Banach valued functions that have finite second moments:

$$L^2_P(\Omega; W(D)) = \Big\{ v : \Omega \to W(D) \mid v \text{ is strongly measurable}, \\ \int_\Omega \|v(\omega, \cdot)\|^2_{W(D)} dP(\omega) < +\infty \Big\}.$$

2.1. State Equations. Soil properties are difficult to measure on the whole spatial domain, therefore the material properties used in the simulation of groundwater flows are usually flawed by uncertainties. There has been recently an increasing interest in the modeling and computational aspects of the uncertainties of the groundwater flow [32, 50, 52] and porous media, see e.g., [18, 26, 54, 38, 53, 58].
We consider the groundwater flow problem in a region $D \subset \mathbb{R}^d$, where the flux is related to the hydraulic head gradient by Darcy's law. We model the uncertainties in the soil by describing the conductivity coefficient κ as a random field denoted $\kappa(\omega, x)$. Similarly, the stochastic forcing term $f(\omega, x)$ models the uncertainty in the sources and sinks (see, e.g. [1, 55, 15] and the references therein). Therefore the hydraulic head p and velocity u are also random fields satisfying the elliptic stochastic partial differential equation (SPDE):

$$\begin{aligned} u(\omega, x) &= -\kappa(\omega, x)\nabla p(\omega, x) && \text{in } \Omega \times D, \\ \nabla \cdot u &= f && \text{in } \Omega \times D, \\ p &= 0 && \text{on } \Omega \times \partial D. \end{aligned} \tag{2.1}$$

In order to write an appropriate weak formulation for (2.1), we need to introduce the Hilbert space (see [27])

$$H(div, D) = \left\{ v \in (L^2(D))^d \mid \nabla \cdot v \in L^2(D) \right\}$$

with the corresponding norm

$$\|v\|_{H(div,D)} = (\|v\|^2_{L^2(D)} + \|\nabla \cdot v\|^2_{L^2(D)})^{1/2}.$$

Currently, numerical methods for Darcy flow consider two different approaches: the first one using the primal, single-phase formulation for pressure, which involves solving a Poisson equation for pressure, and the second one using a mixed, two-phase formulation, with velocity and pressure as the variables of interest.
We will now make the following assumptions concerning the abstract state equations given by (2.1):

A_1) the solution u, p to (2.1) has realizations in the Banach spaces $H(div, D)$ and $L^2(D)$ respectively, i.e., $u(\omega, \cdot) \in H(div, D)$, $p(\omega, \cdot) \in L^2(D)$ almost surely and $\forall \omega \in \Omega$

$$\|u(\omega, \cdot)\|_{H(div,D)} + \|p(\omega, \cdot)\|_{L^2(D)} \leq C\|f(\omega, \cdot)\|_{L^2(D)}$$

where C is a constant independent of the realization $\omega \in \Omega$.

A_2) the forcing term $f \in L^2_P(\Omega; L^2(D))$ is such that the solution u, p is unique and bounded in $L^2_P(\Omega; H(div, D))$ and $L^2_P(\Omega; L^2(D))$ respectively.

The linear elliptic SPDE (2.1) with $\kappa(\omega, \cdot)$ uniformly bounded and coercive, i.e., there exists $\kappa_{min} > 0$ and $\kappa_{max} < \infty$ such that

$$P\left[\omega \in \Omega : \; \kappa_{min} \leq \kappa(\omega, x) \leq \kappa_{max} \, \forall x \in \overline{D}\right] = 1, \tag{2.2}$$

and $f(\omega, \cdot)$ square integrable with respect to P, satisfies assumptions A_1 and A_2 (see [4, 49]).
We shall assume that D is a bounded and open subset of $\mathbb{R}^d$, either with smooth boundary (of class C^2 for instance) or convex. This implies that for every $f \in L^2_P(\Omega; L^2(D))$, problem (2.1) has a unique solution $(u, p) \in L^2_P(\Omega; H(div, D)) \times$

$L^2_P(\Omega; L^2(D))$.
We will denote the expected value of a random variable $X(\omega)$ with probability density function (p.d.f.) ρ by

$$\mathbb{E}\left[X\right] = \int_\Omega X(\omega)dP(\omega) = \int_{\mathbb{R}} x\rho(x)dx. \tag{2.3}$$

The usual multiplication by test functions $v \in H(div, D)$ and $w \in L^2(D)$ and subsequent application of Green's Theorem in the system (2.1) yield the standard weak mixed formulation, namely: find $u(\omega, x) \in L^2_P(\Omega; H(div, D))$ and $p(\omega, x) \in L^2_P(\Omega; L^2(D))$ such that

$$\begin{aligned} &\mathbb{E}\Big[\big(\kappa^{-1}u, v\big)_{(L^2(D))^d} - \big(p, \nabla\cdot v\big)_{L^2(D)}\Big] = 0, \quad \forall v \in H(div, D) \\ &\mathbb{E}\Big[\big(\nabla\cdot u, w\big)_{L^2(D)}\Big] = \mathbb{E}\Big[\big(f, w\big)_{L^2(D)}\Big], \quad \forall w \in L^2(D). \end{aligned} \tag{2.4}$$

Throughout the rest of this chapter, for simplicity of notation, the inner product in $L^2(D)$ or $(L^2(D))^d$ will be denoted by $(\cdot,\cdot)$.

3. Generalized stochastic inverse problems. First we define the admissible set of conductivity coefficients given by

$$\mathcal{A}_{ad} = \left\{\kappa \in L^\infty(\Omega; L^\infty(D)) \mid \kappa(\omega, x) \text{ satisfies } (2.2)\right\}, \tag{3.1}$$

then given $\kappa \in \mathcal{A}_{ad}$ let the admissible set of states and controls be defined as

$$\mathcal{B}_{ad} = \Big\{(u, p, f) \mid u \in L^2_P(\Omega; H(div, D)), p \in L^2_P(\Omega; L^2(D)), f \in L^2_P(\Omega; L^2(D))\Big\}. \tag{3.2}$$

Finally, given $f \in L^2_P(\Omega; L^2(D))$ let the admissible set of states and coefficients be described as

$$\mathcal{C}_{ad} = \Big\{(u, p, \kappa) \mid u \in L^2_P(\Omega; H(div, D)), p \in L^2_P(\Omega; L^2(D)), \kappa \in \mathcal{A}_{ad}\Big\}. \tag{3.3}$$

We also introduce the stochastic target functions $\overline{p} \in L^2_P(\Omega; L^2(D))$ a given possible perturbed observation of the pressure, and $\overline{u} \in L^2_P(\Omega; H(div, D))$ a given possible perturbed observation of the Darcy velocity.

3.1. Stochastic optimal control problems. In this section we consider a general class of minimization problems for solving the stochastic inverse problem for the random forcing function $f(\omega, x)$ and the solution $(u(\omega, x), p(\omega, x))$ satisfying a.s. (2.1). Here we assume as given the input random process $\kappa \in \mathcal{A}_{ad}$ and the targets $\overline{p} \in L^2_P(\Omega; L^2(D))$ and $\overline{u} \in L^2_P(\Omega; H(div, D))$ and we want to recover (u^*_J, p^*_J, f^*_J) such that

$$(u^*_J, p^*_J, f^*_J) = \inf_{(u,p,f)\in\mathcal{B}_{ad}} \left\{J(u, p, f) : \text{ subject to } (2.1)\right\} \tag{3.4}$$

where $J(u, p, f)$ is a given stochastic functional constructed to track the desired random fields $(\overline{u}, \overline{p})$ or the statistical quantities of interest (QoI) of such stochastic functions. This leads to the following definition.

DEFINITION 3.1 (Stochastic optimal control). *A 3-tuple $(u_J^*, p_J^*, f_J^*) \in \mathcal{B}_{ad}$ satisfying (2.1) a.s., for which the infimum in (3.4) is attained is called the stochastic optimal solution and the control f_J^* is referred as stochastic optimal control.* In what follows we will describe two functionals, denoted $J_1(u,p,f)$ and $J_2(u,p,f)$, used to solve stochastic optimal control problems. The first functional, defined in (3.5), is based on the standard classical approach based on stochastic least squares approximation. The second functional, defined in (3.8), uses statistical tracking objectives and is easily generalizable for higher order moments, similarly to (3.20). We will derive the corresponding adjoint equations, state the necessary conditions for existence and uniqueness of the stochastic optimal solution and prove the necessary conditions for optimality.

3.1.1. The optimal control problem using stochastic least squares minimization. For $\kappa \in \mathcal{A}_{ad}$ given data, we consider the following optimal control problem associated with a stochastic elliptic boundary value problem:

(P.1) *Minimize the cost functional*

$$J_1(u,p,f) = \mathbb{E}\Big[\frac{1}{2}\|u(\omega,\cdot) - \overline{u}(\omega,\cdot)\|^2_{(L^2(D))^d} + \frac{1}{2}\|p(\omega,\cdot) - \overline{p}(\omega,\cdot)\|^2_{L^2(D)}\Big] + \mathbb{E}\Big[\frac{\alpha}{2}\|f(\omega,\cdot)\|^2_{L^2(D)}\Big], \tag{3.5}$$

on all $(u,p,f) \in \mathcal{B}_{ad}$ subject to the stochastic mixed state equations (2.1).

Using standard techniques (see e.g. [40, 41, 7, 9, 48, 31, 30, 29]) one can prove that the problem (3.5)-(2.1) has a unique optimal solution that is characterized by a maximum principle type result.
We introduce the co-state elliptic equations, written in weak mixed form:

$$\begin{aligned} \mathbb{E}\left[(\kappa^{-1}q, v) - (z, \nabla\cdot v)\right] &= -\mathbb{E}\left[(u-\overline{u}, v)\right], && \forall v \in H(div, D),\\ \mathbb{E}\left[(\nabla\cdot q, w)\right] &= \mathbb{E}\left[(p-\overline{p}, w)\right], && \forall w \in L^2(D). \end{aligned} \tag{3.6}$$

We now state the necessary conditions for optimality in problem (P.1).

PROPOSITION 3.2. *$(\widehat{u}, \widehat{p}, \widehat{f})$ is the unique optimal solution in problem (3.5)-(2.4) if and only if there exists a co-state $(q,z) \in L^2_P(\Omega; H(div,D)) \times L^2_P(\Omega; L^2(D))$ such that $(\widehat{u}, \widehat{p}, \widehat{f}, q, z)$ satisfies the following optimality conditions:*

$$\begin{cases} \mathbb{E}\left[(\kappa^{-1}\widehat{u}, v) - (\widehat{p}, \nabla\cdot v)\right] = 0, & \forall v \in H(div, D)\\ \mathbb{E}\left[(\nabla\cdot \widehat{u}, w)\right] = \mathbb{E}\left[(\widehat{f}, w)\right], & \forall w \in L^2(D)\\ \mathbb{E}\left[(\kappa^{-1}q, v) - (z, \nabla\cdot v)\right] = -\mathbb{E}\left[(\widehat{u}-\overline{u}, v)\right], & \forall v \in H(div, D),\\ \mathbb{E}\left[(\nabla\cdot q, w)\right] = \mathbb{E}\left[(\widehat{p}-\overline{p}, w)\right], & \forall w \in L^2(D)\\ \mathbb{E}\left[(z+\alpha\widehat{f}, f_s - \widehat{f})\right] \geq 0, & \forall (\widehat{u}, \widehat{p}, f_s) \in \mathcal{B}_{ad}. \end{cases} \tag{3.7}$$

The proof of this result follows in similar manner with the next result, Theorem 1. We note that it is possible to solve the coupled optimality system in one-shot, see e.g. [40].

3.1.2. The optimal control problem utilizing statistical tracking objectives. Now we aim at matching expected values, i.e., we consider the following problem:

(P.2) *Minimize the cost functional*

$$
(3.8)\quad J_2(u,p,f) = \frac{1}{2}\|\mathbb{E}u(\cdot,x) - \mathbb{E}\overline{u}(\cdot,x)\|^2_{(L^2(D))^d} + \frac{1}{2}\|\mathbb{E}p(\cdot,x) - \mathbb{E}\overline{p}(\cdot,x)\|^2_{L^2(D)} + \frac{\alpha}{2}\int_D \mathbb{E}f^2(\cdot,x)dx,
$$

on all $(u,p,f) \in \mathcal{B}_{ad}$ subject to the stochastic mixed state equations (2.1).

REMARK 3.1. *Note that we have*

$$
\int_D \Big[\mathbb{E}u(\cdot,x) - \mathbb{E}\overline{u}(\cdot,x)\Big]^2 dx \le \mathbb{E}\Big(\|u-\overline{u}\|^2_{L^2(D)}\Big),
$$
$$
\int_D \Big[\mathbb{E}p(\cdot,x) - \mathbb{E}\overline{p}(\cdot,x)\Big]^2 dx \le \mathbb{E}\Big(\|p-\overline{p}\|^2_{L^2(D)}\Big),
$$

which justifies the functional (3.8).

THEOREM 3.3. *The 3-tuple $(\widetilde{u},\widetilde{p},\widetilde{f})$ is the unique optimal solution in problem (3.8)-(2.4) if and only if there exists a co-state $(q,z) \in L^2_P(\Omega;H(div,D)) \times L^2_P(\Omega;L^2(D))$ such that $(\widetilde{u},\widetilde{p},\widetilde{f},q,z)$ satisfies the following optimality conditions:*

$$
(3.9)\quad \begin{cases}
\mathbb{E}\left[(\kappa^{-1}\widetilde{u},v) - (\widetilde{p},\nabla\cdot v)\right] = 0, & \forall v \in H(div,D)\\
\mathbb{E}\left[(\nabla\cdot\widetilde{u},w)\right] = \mathbb{E}\Big[(\widetilde{f},w)\Big], & \forall w \in L^2(D)\\
\mathbb{E}\left[(\kappa^{-1}q,v) - (z,\nabla\cdot v)\right] = -\mathbb{E}\left[(\mathbb{E}\widetilde{u} - \mathbb{E}\overline{u},v)\right], & \forall v \in H(div,D),\\
\mathbb{E}\left[(\nabla\cdot q,w)\right] = \mathbb{E}\left[(\mathbb{E}\widetilde{p} - \mathbb{E}\overline{p},w)\right], & \forall w \in L^2(D)\\
\mathbb{E}\Big[(z+\alpha\widetilde{f}, f_s - \widetilde{f})\Big] \ge 0, & \forall(\widetilde{u},\widetilde{p},f_s) \in \mathcal{B}_{ad}.
\end{cases}
$$

Proof. The sensitivity equations corresponding to the state equations (3.6) are

$$
(3.10)\quad \begin{cases}
\mathbb{E}\left[(\kappa^{-1}u_s,v) - (p_s,\nabla\cdot v)\right] = 0, & \forall v \in H(div,D)\\
\mathbb{E}\left[(\nabla\cdot u_s,w)\right] = \mathbb{E}\left[(f_s,w)\right], & \forall w \in L^2(D),
\end{cases}
$$

where $f_s \in L^2_P(\Omega,L^2(D))$, $p_s \in L^2_P(\Omega,L^2(D))$ and $u_s \in L^2_P(\Omega,H(div,D))$. Then the optimality condition for problem (3.8) writes

$$
(3.11)\quad \begin{aligned}
0 &\le \frac{dJ_2(u|_{\widetilde{f}},p|_{\widetilde{f}},\widetilde{f})}{df}f_s \equiv \frac{dJ_2(\widetilde{u},\widetilde{p},\widetilde{f})}{d(u,p,f)}(u_s,p_s,f_s)\\
&= \int_D \mathbb{E}[u_s(\cdot,x)]\mathbb{E}[\widetilde{u}(\cdot,x) - \overline{u}(\cdot,x)]dx + \int_D \mathbb{E}[p_s(\cdot,x)]\mathbb{E}[\widetilde{p}(\cdot,x) - \overline{p}(\cdot,x)]dx\\
&\quad + \alpha\int_D \mathbb{E}[\widetilde{f}f_s]dx\\
&= \int_D \mathbb{E}\Big[u_s(\cdot,x)\mathbb{E}[\widetilde{u}(\cdot,x) - \overline{u}(\cdot,x)]\Big]dx + \int_D \mathbb{E}\Big[p_s(\cdot,x)\mathbb{E}[\widetilde{p}(\cdot,x) - \overline{p}(\cdot,x)]\Big]dx\\
&\quad + \alpha\int_D \mathbb{E}\Big[\widetilde{f}f_s\Big]dx \qquad \text{(since } \mathbb{E}[\widetilde{u}(\cdot,x) - \overline{u}(\cdot,x)] \text{ is deterministic)}\\
&= \int_D \mathbb{E}\Big[-u_s(\cdot,x)\kappa^{-1}q + z\nabla\cdot u_s\Big]dx + \int_D \mathbb{E}\Big[p_s(\cdot,x)\nabla\cdot q\Big]dx + \alpha\int_D \mathbb{E}\Big[\widetilde{f}f_s\Big]dx\\
&\text{(by (3.9) with } v = u_s)
\end{aligned}
$$

$$= \mathbb{E}\left[\int_D -u_s(\cdot,x)\kappa^{-1}q + z\nabla\cdot u_s dx\right] + \mathbb{E}\left[\int_D p_s(\cdot,x)\nabla\cdot q dx\right] + \alpha\mathbb{E}\left[\int_D \widetilde{f} f_s dx\right]$$

(by Fubini's theorem)

$$= \mathbb{E}\left[\int_D -u_s(\cdot,x)\kappa^{-1}q dx + \int_D \nabla\cdot u_s z dx\right] + \mathbb{E}\left[\int_D \kappa^{-1}u_s(\cdot,x)q dx\right]$$

$$+ \mathbb{E}\left[\int_D \alpha\widetilde{f} f_s dx\right] \qquad \text{(by (3.10))}$$

$$= \mathbb{E}\left[\int_D f_s z dx\right] + \mathbb{E}\left[\int_D \alpha\widetilde{f} f_s dx\right]$$

(by (3.10))

$$= \mathbb{E}\left[\int_D \left(z + \alpha\widetilde{f}\right) f_s dx\right]$$

$$= \mathbb{E}\left[\left(z + \alpha\widetilde{f}, f_s\right)\right], \qquad \forall (u_s, p_s, f_s) \in \operatorname{Tan}\mathcal{B}_{ad}\big(\widetilde{u}, \widetilde{p}, \widetilde{f}\big), \quad z + \alpha\widetilde{f} \in \mathrm{N}_{\mathcal{B}_{ad}},$$

where we have used the fact that $\mathbb{E}\left[\widetilde{u}(\cdot,x) - \overline{u}(\cdot,x)\right]$ is a deterministic quantity, the adjoint equations (3.9), Fubini's theorem, the sensitivity equation (3.10) and the definition of normal cone. Here $\operatorname{Tan}\mathcal{B}_{ad}$ denotes the tangent cone, while $\mathrm{N}_{\mathcal{B}_{ad}}$ is the normal cone (see [16]). ▯

The necessary and sufficient conditions (3.9) resemble the optimality system (3.7), the difference is only in the adjoint equations which have a deterministic right-hand side. Nevertheless, the adjoint variables (q, z) are still stochastic quantities.

3.2. Stochastic parameter identification problems. We also study the identification of the coefficient κ in the stochastic boundary value problem (2.1). In the deterministic case, for the direct problem, where κ is given, the existence and uniqueness results are well known, see e.g. [36]. The linear deterministic inverse problem related to (2.1) has been studied in e.g. [7, 48], for the nonlinear deterministic see e.g. [8].

For the identification problem, we are given possible perturbed observations $\overline{u}$, $\overline{p}$ corresponding to the state variables u, respectively p, and we must determine κ in (2.1) such that $u(\kappa) = \overline{u}$ and $p(\kappa) = \overline{p}$ in $\Omega \times D$. Of course, such a κ may not exist.

3.2.1. Parameter identification using stochastic least squares minimization. The least squares approach leads us to the minimization problem:

(P.3) *Minimize the cost functional*

$$J_3(u,p,\kappa) = \mathbb{E}\left[\frac{1}{2}\|u(\omega,\cdot) - \overline{u}(\omega,\cdot)\|^2_{(L^2(D))^d} + \frac{1}{2}\|p(\omega,\cdot) - \overline{p}(\omega,\cdot)\|^2_{L^2(D)}\right] + \mathbb{E}\left[\frac{\beta}{2}\|\kappa(\omega,\cdot)\|^2_{L^2(D)}\right], \tag{3.12}$$

on all $(u,p,\kappa) \in \mathcal{C}_{ad}$ subject to the stochastic mixed state equations (2.1).

We introduce the co-state elliptic equations for this problem (P.3):

$$\begin{cases} \mathbb{E}\Big[((\kappa^*)^{-1}q, v) - (\eta, \nabla\cdot v)\Big] = \mathbb{E}\Big[-(u^* - \overline{u}, v)\Big], & \forall v \in H(div, D) \\ \mathbb{E}\Big[\big(\nabla\cdot q, w\big)\Big] = \mathbb{E}\Big[(p^* - \overline{p}, w)\Big], & \forall w \in L^2(D). \end{cases} \tag{3.13}$$

THEOREM 3.4. *Let (u^*, p^*, κ^*) be an optimal solution in problem (3.12)-(2.4). Then there exists a co-state $(q, \eta) \in L^2_P(\Omega; H(div, D)) \times L^2_P(\Omega; L^2(D))$ such that $(u^*, p^*, \kappa^*, q, \eta)$ satisfies the following optimality conditions:*

$$
(3.14)\qquad \begin{cases}
\mathbb{E}\Big[((\kappa^*)^{-1}u^*, v) - (p^*, \nabla\cdot v)\Big] = 0, & \forall v \in H(div, D)\\
\mathbb{E}\Big[(\nabla\cdot u^*, w)\Big] = \mathbb{E}\Big[(f, w)\Big], & \forall w \in L^2(D)\\
\mathbb{E}\Big[((\kappa^*)^{-1}q, v) - (\eta, \nabla\cdot v)\Big] = \mathbb{E}\Big[-(u^* - \overline{u}, v)\Big], & \forall v \in H(div, D)\\
\mathbb{E}\Big[\Big(\nabla\cdot q, w\Big)\Big] = \mathbb{E}\Big[(p^* - \overline{p}, w)\Big], & \forall w \in L^2(D)\\
\kappa^*(\omega, x) = \max\{\kappa_{\min}, \min\{\frac{1}{\beta}(\kappa^*)^{-2}u^*(\omega, x)q(\omega, x), \kappa_{\max}\}\},
\end{cases}
$$

a.e. in $\Omega \times D$.

Proof. The sensitivity equations are

$$
(3.15)\qquad \begin{cases}
\mathbb{E}\Big[\Big((\kappa^*)^{-1}u_s - (\kappa^*)^{-2}\kappa_s u^*, v\Big) - (p_s, \nabla\cdot v)\Big] = 0, & \forall v \in H(div, D)\\
\mathbb{E}\Big[(\nabla\cdot u_s, w)\Big] = 0, & \forall w \in L^2(D),
\end{cases}
$$

where $(u_s, p_s, \kappa_s) \in \operatorname{Tan}\mathcal{C}_{ad}\big(u^*, p^*, \kappa^*\big)$.

Let $\mathcal{S}_{ad} = \{(u, p, \kappa) \in \mathcal{C}_{ad} : (u, p, \kappa)$ satisfy the state equations (2.1)$\}$ be set of admissible states and parameters to problem (3.12). We introduce the tangential cone to the set $\mathcal{S}_{ad}$ at $(u, p, \kappa) \in \mathcal{S}_{ad}$

$$
\begin{aligned}
\operatorname{Tan}\mathcal{S}_{ad}(u, p, \kappa) = \{&(u_s, p_s, \kappa_s) \text{ which satisfy the sensitivity equations (3.15)},\\
&u_s \in L^2_P(\Omega; H(div, D)), p_s \in L^2_P(\Omega; L^2(D)), \kappa_s \in \operatorname{Tan}\mathcal{A}_{ad}\}.
\end{aligned}
\tag{3.16}
$$

Recall that if

$$
J(u^*, p^*, \kappa^*) = \inf_{(u,p,\kappa)\in\mathcal{S}_{ad}} J(u, p, \kappa)
$$

and the functional $J(u, p, \kappa)$ is Gâteaux differentiable, then necessarily

$$
\frac{dJ(u^*, p^*, \kappa^*)}{d(u, p, \kappa)}(u_s, p_s, \kappa_s) \geq 0 \text{ for all } (u_s, p_s, \kappa_s) \in \operatorname{Tan}\mathcal{S}_{ad}(u^*, p^*, \kappa^*),
\tag{3.17}
$$

where $\frac{dJ(u^*,p^*,\kappa^*)}{d(u,p,\kappa)} \equiv \frac{dJ(u(\kappa^*),p(\kappa^*),\kappa^*)}{d\kappa}$ stands for the Gâteaux derivative of J at $(u^*, p^*, \kappa^*) \in \mathcal{S}_{ad}$, and $(u^*, p^*, \kappa^*) \equiv (u(\kappa^*), p(\kappa^*), \kappa^*)$. Applying the optimum principle given by (3.17) it follows that the optimality condition for problem (3.12) writes

$$
\begin{aligned}
0 &\leq \frac{dJ_3(u(\kappa^*), p(\kappa^*), \kappa^*)}{d\kappa}\kappa_s \equiv \frac{dJ_3(u^*, p^*, \kappa^*)}{d(u, p, \kappa)}(u_s, p_s, \kappa_s)\\
&= \mathbb{E}\Big[\int_D u_s(\cdot, x)\Big(u^*(\cdot, x) - \overline{u}(\cdot, x)\Big)dx\Big] + \mathbb{E}\Big[\int_D p_s(\cdot, x)\Big(p^*(\cdot, x) - \overline{p}(\cdot, x)\Big)dx\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+\mathbb{E}\Big[\beta\int_D \kappa^*(\cdot,x)\kappa_s(\cdot,x)dx\Big]\\
&=\mathbb{E}\Big[-\int_D u_s(\cdot,x)(\kappa^*)^{-1}(\cdot,x)q(\cdot,x)+\eta(\cdot,x)\nabla\cdot u_s(\cdot,x)dx\Big]\\
&+\mathbb{E}\Big[\int_D p_s(\cdot,x)\nabla\cdot q(\cdot,x)dx\Big]+\beta\mathbb{E}\Big[\int_D \kappa^*(\cdot,x)\kappa_s(\cdot,x)dx\Big]\\
&\qquad\text{(by (3.13) with } v=u_s \text{ and } w=p_s)\\
&=\mathbb{E}\Big[-\int_D u_s(\cdot,x)(\kappa^*)^{-1}(\cdot,x)q(\cdot,x)dx\Big]+\mathbb{E}\Big[\int_D p_s(\cdot,x)\nabla\cdot q(\cdot,x)dx\Big]\\
&+\beta\mathbb{E}\Big[\int_D \kappa^*(\cdot,x)\kappa_s(\cdot,x)dx\Big]\qquad\text{(by (3.15) with } w=\eta)\\
&=\mathbb{E}\Big[-\int_D(\kappa^*)^{-2}(\cdot,x)\kappa_s(\cdot,x)u^*(\cdot,x)q(\cdot,x)dx-\int_D p_s(\cdot,x)\nabla\cdot q(\cdot,x)dx\Big]\\
&+\mathbb{E}\Big[\int_D p_s(\cdot,x)\nabla\cdot q(\cdot,x)dx\Big]+\beta\mathbb{E}\Big[\int_D \kappa^*(\cdot,x)\kappa_s(\cdot,x)dx\Big]\qquad\text{(by (3.15) with } v=q)\\
&=\mathbb{E}\Big[\int_D\Big(-(\kappa^*)^{-2}(\cdot,x)u^*(\cdot,x)q(\cdot,x)+\beta\kappa^*(\cdot,x)\Big)\kappa_s(\cdot,x)\Big],
\end{aligned}
$$

for all $(u_s,p_s,\kappa_s)\in \operatorname{Tan}\mathcal{B}_{ad}\big(u^*,p^*,\kappa^*\big)$. Here we have used the adjoint equations (3.13), the sensitivity equations (3.15). $\square$

3.2.2. Parameter identification utilizing statistical tracking objectives. For the identification problem matching *expected values*, given a possible perturbed observation $(\overline{u},\overline{p})$ corresponding to the state variables u, p, we seek κ in (2.1) such that $\mathbb{E}u(\kappa)=\mathbb{E}\overline{u}$ and $\mathbb{E}p(\kappa)=\mathbb{E}\overline{p}$ in D. Therefore we consider the problem:

(P.4) *Minimize the cost functional*

$$
J_4(u,p,\kappa)=\frac{1}{2}\int_D\big[\mathbb{E}u(\cdot,x)-\mathbb{E}\overline{u}(\cdot,x)\big]^2dx+\frac{1}{2}\int_D\big[\mathbb{E}p(\cdot,x)-\mathbb{E}\overline{p}(\cdot,x)\big]^2dx
+\frac{\beta}{2}\int_D\mathbb{E}\kappa^2(\cdot,x)dx, \tag{3.18}
$$

on all $(u,p,\kappa)\in\mathcal{C}_{ad}$ subject to the stochastic state equations (2.1).

THEOREM 3.5. *Let $(\mathring{u},\mathring{p},\mathring{\kappa})$ be an optimal solution in problem (2.1) and (3.18). Then there exists a co-state $(q,\eta)\in L^2_P(\Omega;H(div,D))\times L^2_P(\Omega;L^2(D))$ such that $(\mathring{u},\mathring{p},\mathring{\kappa},q,\eta)$ satisfies the following optimality conditions:*

$$
\begin{cases}
\mathbb{E}\Big[(\mathring{\kappa}^{-1}\mathring{u},v)-(\mathring{p},\nabla\cdot v)\Big]=0, & \forall v\in H(div,D)\\
\mathbb{E}\Big[(\nabla\cdot\mathring{u},w)\Big]=\mathbb{E}\Big[(f,w)\Big], & \forall w\in L^2(D)\\
\mathbb{E}\Big[(\mathring{\kappa}^{-1}q,v)-(\eta,\nabla\cdot v)\Big]=\mathbb{E}\Big[-(\mathbb{E}\mathring{u}-\mathbb{E}\overline{u},v)\Big], & \forall v\in H(div,D)\\
\mathbb{E}\Big[\Big(\nabla\cdot q,w\Big)\Big]=\mathbb{E}\Big[(\mathbb{E}\mathring{p}-\mathbb{E}\overline{p},w)\Big], & \forall w\in L^2(D)\\
\mathring{\kappa}(\omega,x)=\max\{\kappa_{\min},\min\{\frac{1}{\beta}(\mathring{\kappa})^{-2}\mathring{u}(\omega,x)q(\omega,x),\kappa_{\max}\}\},\ a.e.\ in\ \Omega\times D.
\end{cases}
\tag{3.19}
$$

Proof. See the proof of Theorem 3.6. $\square$

For the problem of matching *covariance*, and/or *higher order moments*, the cost functional used in problem (3.18) can be generalized as follows. Assume we are interested in $\mathcal{L}$-order moments, and $f \in L_P^{\mathcal{L}}(\Omega; L^{2\mathcal{L}-2}(D))$, then

(P.5) *Minimize the cost functional*

$$J_5(u,p,\kappa) = \sum_{\ell=1}^{\mathcal{L}} \frac{\alpha_{u,\ell}}{2\ell} \int_D \left[\mathbb{E}u^\ell(\cdot,x) - \mathbb{E}\overline{u}^\ell(\cdot,x)\right]^2 dx + \sum_{\ell=1}^{\mathcal{L}} \frac{\alpha_{p,\ell}}{2\ell} \int_D \left[\mathbb{E}p^\ell(\cdot,x) - \mathbb{E}\overline{p}^\ell(\cdot,x)\right]^2 dx + \frac{\beta}{2} \int_D \mathbb{E}\kappa^2(\cdot,x) dx, \tag{3.20}$$

on all $(u,p,\kappa) \in \mathcal{C}_{ad}$ subject to the stochastic state equations (2.1).

We introduce the co-state elliptic equations for this problem (P.5):

$$\begin{cases} \mathbb{E}\Big[((\kappa^*)^{-1}q, v) - (\eta, \nabla\cdot v)\Big] = -\mathbb{E}\Big[\sum_{\ell=1}^{\mathcal{L}} \alpha_{u,\ell}(u^*)^{\ell-1}(\mathbb{E}(u^*)^\ell - \mathbb{E}\overline{u}^\ell, v)\Big], \\ \mathbb{E}\Big[\Big(\nabla\cdot q, w\Big)\Big] = \mathbb{E}\Big[\sum_{\ell=1}^{\mathcal{L}} \alpha_{p,\ell}(p^*)^{\ell-1}(\mathbb{E}(p^*)^\ell - \mathbb{E}\overline{p}^\ell, w)\Big], \end{cases} \tag{3.21}$$

for all $v \in H(div, D), w \in L^2(D)$.

THEOREM 3.6. *Let (u^*, p^*, κ^*) be an optimal solution in problem (3.20)-(2.4). Then there exists a co-state $(q,\eta) \in L_P^2(\Omega; H(div,D)) \times L_P^2(\Omega; L^2(D))$ such that $(u^*, p^*, \kappa^*, q, \eta)$ satisfies the following optimality conditions:*

$$\begin{cases} \mathbb{E}\Big[((\kappa^*)^{-1}u^*, v) - (p^*, \nabla\cdot v)\Big] = 0, \quad \forall v \in H(div, D) \\ \mathbb{E}\Big[(\nabla\cdot u^*, w)\Big] = \mathbb{E}\Big[(f, w)\Big], \quad \forall w \in L^2(D) \\ \mathbb{E}\Big[((\kappa^*)^{-1}q, v) - (\eta, \nabla\cdot v)\Big] = -\mathbb{E}\Big[\sum_{\ell=1}^{\mathcal{L}} \alpha_{u,\ell}(u^*)^{\ell-1}(\mathbb{E}(u^*)^\ell - \mathbb{E}\overline{u}^\ell, v)\Big], \\ \qquad \forall v \in H(div, D) \\ \mathbb{E}\Big[\Big(\nabla\cdot q, w\Big)\Big] = \mathbb{E}\Big[\sum_{\ell=1}^{\mathcal{L}} \alpha_{p,\ell}(p^*)^{\ell-1}(\mathbb{E}(p^*)^\ell - \mathbb{E}\overline{p}^\ell, w)\Big], \quad \forall w \in L^2(D) \\ \kappa^*(\omega,x) = \max\{\kappa_{\min}, \min\{\frac{1}{\beta}(\kappa^*)^{-2}u^*(\omega,x)q(\omega,x), \kappa_{\max}\}\}, \ \textit{a.e. in } \Omega\times D. \end{cases} \tag{3.22}$$

Proof. The sensitivity equations are

$$\begin{cases} \mathbb{E}\Big[\Big((\kappa^*)^{-1}u_s - (\kappa^*)^{-2}\kappa_s u^*, v\Big) - (p_s, \nabla\cdot v)\Big] = 0, \qquad \forall v \in H(div, D) \\ \mathbb{E}\Big[(\nabla\cdot u_s, w)\Big] = 0, \qquad \forall w \in L^2(D), \end{cases} \tag{3.23}$$

where $(u_s, p_s, \kappa_s) \in \operatorname{Tan} \mathcal{C}_{ad}\big(u^*, p^*, \kappa^*\big)$. Applying the optimum principle given by (3.17) it follows that the optimality condition for problem (3.20) writes

$$
\begin{aligned}
0 &\le \frac{dJ_5(u(\kappa^*), p(\kappa^*), \kappa^*)}{d\kappa}\kappa_s \equiv \frac{dJ_5(u^*, p^*, \kappa^*)}{d(u, p, \kappa)}(u_s, p_s, \kappa_s) \\
&= \sum_{\ell=1}^{\mathcal{L}} \int_D \alpha_{u,\ell} \mathbb{E}\Big[u_s(\cdot, x)(u^*)^{\ell-1}(\cdot, x)\Big] \mathbb{E}\Big[(u^*)^{\ell}(\cdot, x) - \overline{u}^{\ell}(\cdot, x)\Big] dx \\
&\quad + \sum_{\ell=1}^{\mathcal{L}} \int_D \alpha_{p,\ell} \mathbb{E}\Big[p_s(\cdot, x)(p^*)^{\ell-1}(\cdot, x)\Big] \mathbb{E}\Big[(p^*)^{\ell}(\cdot, x) - \overline{p}^{\ell}(\cdot, x)\Big] dx \\
&\quad + \beta \int_D \mathbb{E}\Big[\kappa^*(\cdot, x)\kappa_s(\cdot, x)\Big] dx \\
&= \sum_{\ell=1}^{\mathcal{L}} \int_D \alpha_{u,\ell} \mathbb{E}\Big[u_s(\cdot, x)(u^*)^{\ell-1}(\cdot, x)\mathbb{E}\Big((u^*)^{\ell}(\cdot, x) - \overline{u}^{\ell}(\cdot, x)\Big)\Big] dx \\
&\quad + \sum_{\ell=1}^{\mathcal{L}} \int_D \alpha_{p,\ell} \mathbb{E}\Big[p_s(\cdot, x)(p^*)^{\ell-1}(\cdot, x)\mathbb{E}\Big((p^*)^{\ell}(\cdot, x) - \overline{p}^{\ell}(\cdot, x)\Big)\Big] dx \\
&\quad + \beta \int_D \mathbb{E}\Big[\kappa^*(\cdot, x)\kappa_s(\cdot, x)\Big] dx \\
&= \int_D \mathbb{E}\Big[-u_s(\cdot, x)(\kappa^*)^{-1}(\cdot, x)q(\cdot, x) + \eta(\cdot, x)\nabla \cdot u_s(\cdot, x)\Big] dx \\
&\quad + \int_D \mathbb{E}\Big[p_s(\cdot, x)\nabla \cdot q(\cdot, x)\Big] dx + \beta \int_D \mathbb{E}\Big[\kappa^*(\cdot, x)\kappa_s(\cdot, x)\Big] dx \\
&\qquad \text{(by (3.21) with } v = u_s \text{ and } w = p_s\text{)} \\
&= \mathbb{E}\Big[\int_D -u_s(\cdot, x)(\kappa^*)^{-1}(\cdot, x)q(\cdot, x) + \eta(\cdot, x)\nabla \cdot u_s(\cdot, x)dx\Big] \\
&\qquad + \beta \mathbb{E}\Big[\int_D \kappa^*(\cdot, x)\kappa_s(\cdot, x)dx\Big] \\
&= \mathbb{E}\Big[-\int_D u_s(\cdot, x)(\kappa^*)^{-1}(\cdot, x)q(\cdot, x)dx\Big] + \mathbb{E}\Big[\int_D p_s(\cdot, x)\nabla \cdot q(\cdot, x)dx\Big] \\
&+ \beta \mathbb{E}\Big[\int_D \kappa^*(\cdot, x)\kappa_s(\cdot, x)dx\Big] \qquad \text{(by (3.23) with } w = \eta\text{)} \\
&= \mathbb{E}\Big[-\int_D (\kappa^*)^{-2}(\cdot, x)\kappa_s(\cdot, x)u^*(\cdot, x)q(\cdot, x)dx - \int_D p_s(\cdot, x)\nabla \cdot q(\cdot, x)dx\Big] \\
&+ \mathbb{E}\Big[\int_D p_s(\cdot, x)\nabla \cdot q(\cdot, x)dx\Big] + \beta \mathbb{E}\Big[\int_D \kappa^*(\cdot, x)\kappa_s(\cdot, x)dx\Big] \qquad \text{(by (3.23) with } v = q\text{)} \\
&= \mathbb{E}\Big[\int_D \Big(-(\kappa^*)^{-2}(\cdot, x)u^*(\cdot, x)q(\cdot, x) + \beta\kappa^*(\cdot, x)\Big)\kappa_s(\cdot, x)\Big],
\end{aligned}
$$

for all $(u_s, p_s, \kappa_s) \in \operatorname{Tan} \mathcal{B}_{ad}\big(u^*, p^*, \kappa^*\big)$. Here we have used the adjoint equations (3.21), the sensitivity equations (3.23). □

4. Error estimates for the parameter identification problems under open-interval parameter and finite-dimensional noise assumptions. For reader's convenience, we provide a brief derivation of error estimates [28] under the following assumptions

(i) (Finite-dimensional noise) $\kappa(\omega, x)$ and $f(x, \omega)$ can be expressed in terms of a finite number of independent random variables with bounded support $y_1(\omega)$, $\ldots$, $y_N(\omega)$.

(ii) The coefficient κ belongs to the open interval $(\kappa_{\min}, \kappa_{\max})$, i.e.,

$$P\big[\omega \in \Omega \,:\; \kappa_{min} < \kappa(\omega, x) < \kappa_{max} \;\; \forall x \in \overline{D}\,\big] = 1. \tag{4.1}$$

Using the finite-dimensional noise assumption(i), the solution u, p of (2.1) depends on the realization $\omega \in \Omega$ through the value taken by the random vector $y(\omega) = \big(y_1(\omega), \ldots, y_N(\omega)\big)$, i.e., $u = u(y(\omega), x), p = p(y(\omega), x)$. Therefore, we can replace the probability space $(\Omega, \mathcal{F}, P)$ with $(\Gamma, \mathcal{B}(\Gamma), \rho(y)dy)$, where $\Gamma = y(\Omega)$ denotes the image of the random variable y, $\mathcal{B}(\Gamma)$ denotes the Borel σ-algebra on Γ, and $\rho(y)dy$ denotes the distribution measure of the vector y with $\rho(y) : \Gamma \to \mathbb{R}_+$ denoting the joint probability density function for y. Then the stochastic domain $\Gamma = \prod_{n=1}^N \Gamma_n$, where $\Gamma_n = y_n(\Omega)$ denotes the image of the random variable y_n, and also $\rho(y) = \prod_{n=1}^N \rho_n(y_n)$, where $\rho_n(y_n) : \Gamma_n \to \mathbb{R}_+$ denotes the probability density function for the random variable y_n.

Denoting by

$$\kappa_1 = \frac{1}{2}\big(\kappa_{\min} + \kappa_{\max}\big), \quad \kappa_2 = \frac{1}{\pi}\big(\kappa_{\max} - \kappa_{\min}\big), \tag{4.2}$$

we have that under the assumption (ii)

$$\kappa(\omega, x) = \kappa_1 + \kappa_2 \arctan(\nu(\omega, x)), \tag{4.3}$$

is a bijection from $\mathbb{R}$ onto $(\kappa_{\min}, \kappa_{\max})$, and yields a re-parametrization of problems (P.3)-(P.5) without any restriction on the new variable ν

$$\min_{(u,p,\kappa)\in\mathcal{C}_{ad}} \{J_i(u, p, \kappa) : \text{subject to } (2.4)\} = \min_{(u,p,\nu)\in\mathcal{C}_{ad}^{\nu}} \{J_i(u, p, \kappa(\nu)) : (2.4), (4.3)\},$$

where

$$\mathcal{C}_{ad}^{\nu} = \Big\{(u, p, \kappa) \mid u \in L_P^2(\Omega; H(div, D)), p \in L_P^2(\Omega; L^2(D)), \kappa \in L^\infty(\Omega) \times L^\infty(D)\Big\}.$$

The optimality systems for problems (P.3)-(P.5) can be now be expressed in terms of ν instead of κ. For $i = 3, 4, 5$, we denote by ν_{J_i} the optimal parameter, u_{J_i}, p_{J_i} the corresponding state variables, and q_{J_i}, η_{J_i} the adjoint states.

In view of (2.3) and the above considerations, the state equations (2.4) are

$$\begin{aligned} &\int_D \Big(\big(\kappa_1 + \kappa_2 \arctan(\nu_{J_i}(y, x))\big)^{-1} u_{J_i}(y, x) \cdot v(x) - p_{J_i}(y, x) \nabla \cdot v(x)\Big) dx = 0, \\ &\int_D \nabla \cdot u_{J_i}(y, x)\, w(x)\, dx = \int_D f(y, x)\, w(x)\, dx, \end{aligned} \tag{4.4}$$

for all $v \in H(div, D), w \in L^2(D)$, ρ-a.e. in Γ. Similarly, the adjoint equations are

(4.5)
$$\int_D \Big(\big((\kappa_1 + \kappa_2 \arctan(\nu_{J_i}(y,x)) \big)^{-1} q_{J_i}(y,x) \cdot v(x) - \eta_{J_i}(y,x) \nabla \cdot v(x) \Big)\, dx$$
$$= \begin{cases} -\displaystyle\int_D \big(u_{J_3}(y,x) - \overline{u}(y,x)\big) v(x)\, dx, \\ -\displaystyle\int_D \big(\mathbb{E} u_{J_4}(\cdot,x) - \mathbb{E}\overline{u}(\cdot,x)\big) v(x)\, dx, \\ -\displaystyle\int_D \sum_{\ell=1}^{\mathcal{L}} \alpha_{u,\ell} (u_{J_5}(y,x))^{\ell-1} \big(\mathbb{E}(u_{J_5}(\cdot,x))^\ell - \mathbb{E}(\overline{u}(\cdot,x))^\ell\big) v(x)\, dx, \end{cases}$$
$$\int_D \nabla \cdot q_{J_i}(y,x)\, w(x)\, dx = \begin{cases} \displaystyle\int_D \big(p_{J_3}(y,x) - \overline{p}(y,x)\big) w(x)\, dx, \\ \displaystyle\int_D \big(\mathbb{E} p_{J_4}(\cdot,x) - \mathbb{E}\overline{p}(\cdot,x)\big) w(x)\, dx, \\ \displaystyle\int_D \sum_{\ell=1}^{\mathcal{L}} \alpha_{p,\ell} p_{J_5}^{\ell-1}(y,x) \big(\mathbb{E} p_{J_5}^\ell(\cdot,x)) - \mathbb{E}\overline{p}^\ell(\cdot,x)\big) w(x)\, dx, \end{cases}$$

and the optimality condition writes

$$\nu_{J_i}(y,x) = \tan\Big(\frac{1}{\kappa_2}\Big(\big(\tfrac{1}{\beta} u_{J_i} \cdot q_{J_i}\big)^{\frac{1}{3}} - \kappa_1\Big)\Big). \tag{4.6}$$

In particular, using (4.3) and (4.6), we have that

$$(\kappa_{\min}, \kappa_{\max}) \ni \kappa(\nu_{J_i}) = \Big(\frac{1}{\beta} u_{J_i} \cdot q_{J_i}\Big)^{\frac{1}{3}} \qquad \text{a.s., a.e.,} \tag{4.7}$$

hence the optimality system for problems (P.3)-(P.5) can be written as

$$\int_D \Big(\big(\frac{1}{\beta} u_{J_i}(y,x) \cdot q_{J_i}(y,x)\big)^{-\frac{1}{3}} u_{J_i}(y,x) \cdot v(x) - p_{J_i}(y,x) \nabla \cdot v(x) \Big) dx = 0,$$
$$\int_D \nabla \cdot u_{J_i}(y,x)\, w(x)\, dx = \int_D f(y,x)\, w(x)\, dx,$$
$$\int_D \Big(\big(\frac{1}{\beta} u_{J_i}(y,x) \cdot q_{J_i}(y,x)\big)^{-\frac{1}{3}} q_{J_i}(y,x) \cdot v(x) - \eta_{J_i}(y,x) \nabla \cdot v(x) \Big)\, dx$$
$$= \begin{cases} -\displaystyle\int_D \big(u_{J_3}(y,x) - \overline{u}(y,x)\big) v(x)\, dx, \\ -\displaystyle\int_D \big(\mathbb{E} u_{J_4}(y,x) - \mathbb{E}\overline{u}(y,x)\big) v(y,x)\, dx, \\ -\displaystyle\int_D \sum_{\ell=1}^{\mathcal{L}} \alpha_{u,\ell} (u_{J_5}(y,x))^{\ell-1} \big(\mathbb{E}(u_{J_5}(y,x))^\ell - \mathbb{E}(\overline{u}(y,x))^\ell\big) v(x)\, dx, \end{cases}$$
$$\int_D \nabla \cdot q_{J_i}(y,x) w(x) dx = \begin{cases} \displaystyle\int_D \big(p_{J_3}(y,x) - \overline{p}(y,x)\big) w(x)\, dx, \\ \displaystyle\int_D \big(\mathbb{E} p_{J_4}(y,x) - \mathbb{E}\overline{p}(\cdot,x)\big) w(x)\, dx, \\ \displaystyle\int_D \sum_{\ell=1}^{\mathcal{L}} \alpha_{p,\ell} p_{J_5}^{\ell-1}(\cdot,x) \big(\mathbb{E} p_{J_5}^\ell(\cdot,x)) - \mathbb{E}\overline{p}^\ell(\cdot,x)\big) w(x) dx. \end{cases}$$

The error estimates we derive make use of the Fink-Rheinboldt theory [19, 20, 21, 22] concerning the approximation of a class of nonlinear problems. For the sake of completeness, we state the relevant results of that theory, specialized to our needs.

The parameter-dependent nonlinear problems considered are of the type

$$F(\theta) \equiv F(\zeta, \lambda) = 0, \tag{4.8}$$

where $F : \Theta = Z \times \Lambda \to \Xi$ is a $C^r, r \geq 1$, Fredholm mapping of index 1 from an open subset S of a real Banach space Θ into another Banach space Ξ. In our context, Z is an infinite-dimensional space whereas $\Lambda \subset \mathbb{R}$ is a one-dimensional parameter space. We are interested in approximations of solutions of the infinite-dimensional problem (4.8) obtained by solving a finite-dimensional (discretized) approximate problem

$$\widetilde{F}(\theta) \equiv \widetilde{F}(\widetilde{\zeta}, \lambda) = 0, \tag{4.9}$$

where $\widetilde{F}$ is a mapping from the discretized space $\widetilde{\Theta} = \widetilde{Z} \times \Lambda$ into another discretized space $\widetilde{\Xi}$.

Suppose we have a point $\theta_0 \in \Theta$ which may not solve the problem (4.8) but instead certainly solves the problem

$$F(\theta) \equiv F(\zeta_0, \lambda_0) = \xi_0 \quad \text{with} \quad \xi_0 := F(\theta_0) \tag{4.10}$$

so that it lies on the solution manifold $\mathcal{M}_{\xi_0}$ of this equation. If $DF(\theta) \in \mathscr{L}(\Theta, \Xi)$ denotes the total Fréchet derivative of F at the point θ, then a point $\theta \in S$ is referred to as a *a regular point* of F if $DF(\theta)$ is surjective; a point $\xi \in \Xi$ is a *regular value* of F if $F^{(-1)}(\xi)$ contains only regular points. If $\xi \in \Xi$ is a regular value of F, then $M_\xi \equiv F^{(-1)}(\xi)$ is an m-dimensional C^r-manifold in X without boundary.

A basis on Λ can serve as a local coordinate system for the manifold $\mathcal{M}_{\xi_0}$ at a point $\theta_0 \in \mathcal{M}_{\xi_0}$ provided that $\ker D_\zeta F(\theta_0) = \{0\}$, where $D_\zeta F(\theta_0) \in \mathscr{L}(Z, Y)$ denotes the Z-partial derivative of F at the point θ_0.

THEOREM 4.1. [20] *Let ξ_0 denote a regular value of F and let $\theta_0 \in M_{\xi_0}$. Suppose that $\Theta = Z \times \Lambda \to \Xi$, $\dim \Lambda = 1$, is a splitting such that $\ker D_\zeta F(\theta_0) = \{0\}$. Set $\theta_0 = (\zeta_0, \lambda_0)$ with $\zeta_0 \in Z$ and $\lambda_0 \in \Lambda$. Then, there exist open neighborhoods $B \subset \Lambda$ of λ_0 and $U \subset \Theta$ of θ_0 and a unique C^r-function $\zeta : B \to Z$ such that $\zeta(\lambda_0) = \zeta_0$ and*

$$\mathcal{M}_{\xi_0} \cap U = \{\theta \in \Theta \,:\, \theta = \big(\zeta(\lambda), \lambda\big),\, \lambda \in B\}.$$

Any $\theta_0 \in S$ where $\ker D_\zeta F(\theta_0) = \{0\}$ is referred to as a *nonsingular point* of F (with respect to the splitting $X = Z \oplus \Lambda$); otherwise, θ_0 is called a *singular point*. Thus, if $\theta_0 \in S$ is a nonsingular point, then $\ker D_\zeta F(\theta_0)$ is an isomorphism of Z onto Ξ.

We next choose a finite-dimensional subspace $\widetilde{Z} \subset Z$, a projection $\widetilde{Q}Z \to \widetilde{Z}$ of Z onto $\widetilde{Z}$, and an isomorphism $J : Z \to \Xi$. With $\widetilde{\Theta} = \widetilde{Z} \times \Lambda$, $\widetilde{\Xi} = J\widetilde{Z}$, and the projection $\widetilde{P} : \Xi \to \widetilde{\Xi}$ specified by $\widetilde{P} = J\widetilde{Q}J^{-1}$, we then define a function $\widetilde{F} : S \to \widetilde{\Xi}$ by $\widetilde{F}(\theta) = \widetilde{P}F(\theta)$, $\theta \in S$. Then, corresponding to (4.9), we have the finite-dimensional (discretized) problem

$$\widetilde{F}(\theta_0) = \widetilde{\xi}_0 \quad \text{with} \quad \widetilde{\xi}_0 := \widetilde{F}(\theta_0). \tag{4.11}$$

THEOREM 4.2. [20] *Suppose that the mapping F is of class C^r, $r \geq 2$, and that the second Fréchet derivative $D^2_\theta F$ is bounded on bounded subsets of its domain S.*

Consider a point $\theta_0 \in S$ such that $\xi_0 = F(\theta_0)$ is a regular value of F and assume that $\ker D_\zeta F(\theta_0) = \{0\}$. *Set $\theta_0 = (\zeta_0, \lambda_0)$, $\lambda_0 \in \Lambda$, and let $\theta : B \subset \Lambda \to S$, $\theta(\lambda) = \big(\zeta(\lambda), \lambda\big)$ denote the solution of (4.8) as given by Theorem 4.1. Set $\widetilde{Z} = Z \cap \widetilde{\Theta}$ and suppose that* $\ker \widetilde{P} D_\zeta F(\theta_0)|_{\widetilde{Z}} = \{0\}$. *Then, there exist a closed ball $B_0 \subset B$ centered at λ_0 and a C^r-function $\widetilde{\zeta} : B_0 \to \widetilde{Z}$ such that $\widetilde{\theta}(\lambda) = \big(\widetilde{\zeta}(\lambda), \lambda\big)$ solves (4.10) for each $\lambda \in B_0$ and*

$$\|\zeta(\lambda) - \widetilde{\zeta}(\lambda)\|_Z \le C \|(I_Z - \widetilde{Q})(\zeta(\lambda) - \zeta_0)\|_Z \quad \forall\, \lambda \in B_0, \tag{4.12}$$

where C is a constant independent of λ.

We focus on the optimality system corresponding to optimal parameter identification problem (P.4). The error analyses for problems (P.3) and (P.5) are very similar. Also, the optimal control problems (P.1) and (P.2) can be treated in a similar, albeit simpler, manner. Thus, for the sake of economy of exposition, from now on we drop the subscript $(\cdot)_{J_4}$ so that, e.g., now $u_{J_4} = u, p_{J_4} = p$, $q_{J_4} = q, \eta_{J_4} = \eta$.

We recast the optimality system (4.4)-(4.6) into a form that fits the above framework so that we can apply Theorem 4.2, without major modifications, to derive an error estimate for solutions of the optimality system for the parameter identification problem (P.4).

We begin by choosing the state space $Z = [L^2_\rho\big(\Gamma; H(div, D)\big) \times L^2_\rho\big(\Gamma; L^2(D)\big)]^2$ and also $\Lambda = \mathbb{R}^+$ with $\zeta = (u, p, q, \eta)$ and $\lambda = 1/\beta$ so that $\Theta = Z \times \Lambda = [L^2_\rho\big(\Gamma; H(div, D)\big) \times L^2_\rho\big(\Gamma; L^2(D)\big)]^2 \times \mathbb{R}^+$. Then, the range space Ξ is the dual space of Z with respect to the standard duality pairing

$$\big(\psi, \varphi\big) = \int_D \int_\Gamma \varphi \psi \rho(y) dy dx \qquad \text{for } \varphi \in Z \text{ and } \psi \in \Xi. \tag{4.13}$$

We also choose the space $S = [L^6_\rho\big(\Gamma; H(div, D) \cap L^6(D)\big) \times L^2_\rho\big(\Gamma; L^2(D)\big)]^2 \times \mathbb{R}^+ \subset \Theta$.

For convenience, we choose to define the function $F(\theta)$ and its derivatives weakly in terms of the duality pairing. Following (4.6) and (4.3), for each $\theta = \big((u, p, q, \eta), \beta\big) \in \Theta$, we denote

$$\nu(\theta) = \tan\Big(\frac{1}{\kappa_2}\Big(\big(\tfrac{1}{\beta} u \cdot q\big)^{\frac{1}{3}} - \kappa_1\Big)\Big) \quad \text{and} \quad \kappa(\theta) = \kappa_1 + \kappa_2 \arctan(\nu(\theta)), \tag{4.14}$$

and define $F(\theta)$ by

$$\begin{aligned}
&\Big(F\big((u, p, q, \eta), \beta\big)\,,\, (\check{u}, \check{p}, \check{q}, \check{\eta})\Big) \\
&= \big(\kappa^{-1}(\theta)\, u\,,\, \check{u}\big) - \big(p, \nabla \cdot \check{u}\big) + \big(\nabla \cdot u - f, \check{p}\big) \\
&\quad + \big(\kappa^{-1}(\theta)\, q\,,\, \check{q}\big) - \big(\eta, \nabla \cdot \check{q}\big) + \big(\mathbb{E}u - \mathbb{E}\overline{u}\,, \check{q}\big), \\
&\quad + \big(\nabla \cdot q - (\mathbb{E}p - \mathbb{E}\overline{p})\,, \check{\eta}\,\big) \qquad \forall\, (\check{u}, \check{p}, \check{q}, \check{\eta}) \in Z
\end{aligned} \tag{4.15}$$

so that the optimality conditions (4.4)-(4.6) for problem (P.4) become: seek $(u, p, q, \eta) \in Z = [L^2_\rho\big(\Gamma; H(div, D)\big) \times L^2_\rho\big(\Gamma; L^2(D)\big)]^2$ such that

$$\Big(F\big((u, p, q, \eta), \beta\big)\,,\, (\check{u}, \check{p}, \check{q}, \check{\eta},)\Big) = 0 \qquad \forall\, (\check{u}, \check{p}, \check{q}, \check{\eta}) \in Z. \tag{4.16}$$

Let $\widetilde{Z} \subset Z$ denote a finite-dimensional subspace. Then, from (4.15), for any $\theta \in \Theta$, the discrete operator $\widetilde{F}(\theta) = \widetilde{P}F(\theta)$ is defined by

$$
\begin{aligned}
&\Big(\widetilde{F}\big((u,p,q,\eta),\beta\big)\,,\,(\check{u},\check{p},\check{q},\check{\eta})\Big) \\
&= \big(\kappa^{-1}(\theta)\,u\,,\,\check{u}\big) - \big(p,\nabla\cdot\check{u}\big) + \big(\nabla\cdot u - f,\check{p}\big) \\
&\quad + \big(\kappa^{-1}(\theta)\,q\,,\,\check{q}\big) - \big(\eta,\nabla\cdot\check{q}\big) + \big(\mathbb{E}u - \mathbb{E}\overline{u}\,,\check{q}\big), \\
&\quad + \big(\nabla\cdot q - (\mathbb{E}p - \mathbb{E}\overline{p})\,,\check{\eta}\big) \qquad \forall\,(\check{u},\check{p},\check{q},\check{\eta}) \in \widetilde{Z}
\end{aligned}
\tag{4.17}
$$

and the discrete problem (4.9) becomes: seek $(\widetilde{u},\widetilde{p},\widetilde{q},\widetilde{\eta}) \in \widetilde{Z} \subset Z$ such that

$$
\Big(\widetilde{F}\big((\widetilde{u},\widetilde{p},\widetilde{q},\widetilde{\eta}),\beta\big)\,,\,(\check{u},\check{p},\check{q},\check{\eta})\Big) = 0 \qquad \forall\,(\check{u},\check{p},\check{q},\check{\eta}) \in \widetilde{Z}. \tag{4.18}
$$

Note that, through the use of the weak definition of the various mappings, the projection operator $\widetilde{Q}$ and the isomorphism J are implicitly induced through the duality pairing. In fact, we have that J is the Riesz mapping connecting $Z = [L^2_\rho\big(\Gamma;H(div,D)\big) \times L^2_\rho\big(\Gamma;L^2(D)\big)]^2$ with its dual space and $\widetilde{Q} : Z \to \widetilde{Z}$ is the projection with respect to the inner product in $[L^2_\rho\big(\Gamma;H(div,D)\big) \times L^2_\rho\big(\Gamma;L^2(D)\big)]^2$.

To derive error estimates by applying Theorem 4.2 to our setting, we first need to verify the assumptions of that theorem and those of Theorem 4.1.

The first partial Fréchet derivative $D_\zeta F(\theta_0) \in \mathcal{L}(\Theta,\Xi)$ is given by, with $\zeta = (u,p,q,\eta)$, $\theta_0 = \big((u_0,p_0,q_0,\eta_0),\beta_0\big)$, $\theta_1 = \big((u_1,p_1,q_1,\eta_1),\beta_1\big)$,

$$
\begin{aligned}
\big(D_\zeta F(\theta_0)\theta_1,\check{\zeta}\big) &= \big(\kappa^{-1}(\theta_0)u_1\,,\check{u}\big) + \big(\widetilde{\kappa}(\theta_0,\theta_1)u_0\,,\check{u}\big) - (p_1,\nabla\cdot\check{u}) + (\nabla\cdot u_1,\check{p}) \\
&\quad + \big(\kappa^{-1}(\theta_0)q_1\,,\check{q}\big) + \big(\widetilde{\kappa}(\theta_0,\theta_1)q_0\,,\check{q}\big) - (\eta_1,\nabla\cdot\check{q}) + (\mathbb{E}u_1,\check{q}) \\
&\quad + (\nabla\cdot q_1 - \mathbb{E}p_1,\check{\eta})
\end{aligned}
\tag{4.19}
$$

for all $\check{\zeta} = (\check{u},\check{p},\check{q},\check{\eta}) \in Z$, where

$$
\widetilde{\kappa}(\theta_0,\theta_1) = -\frac{1}{3\beta_0}\kappa^{-4}(\theta_0)\big(u_1\cdot q_0 + u_0\cdot q_1\big).
$$

The Fréchet derivative $DF(\theta_0) \in \mathcal{L}(\Theta,\Xi)$ is then given by

$$
\big(DF(\theta_0)\theta_1,\check{\zeta}\big) = \big(D_\zeta F(\theta_0)\theta_1,\check{\zeta}\big) + \big(D_\beta F(\theta_0)\theta_1,\check{\zeta}\big),
$$

where

$$
\big(D_\beta F(\theta_0)\theta_1,\check{\zeta}\big) = \frac{\beta_1}{3\beta_0^2}\Big(\kappa^{-4}(\theta_0)u_0\cdot q_0\,u_0,\check{u}\Big) + \frac{\beta_1}{3\beta_0^2}\Big(\kappa^{-4}(\theta_0)u_0\cdot q_0\,q_0,\check{q}\Big).
$$

The second Fréchet derivative $D^2_\theta F(\theta) \in \mathcal{L}(\Theta\times\Theta,\Xi)$, with $\theta_2 = \big((u_2,p_2,q_2,\eta_2),\beta_2\big)$, is given by

$$
\begin{aligned}
\big(D^2_\theta F(\theta_0)\theta_1\theta_2\,,\,\check{\zeta}\big) &= \big(\chi(\theta_0,\theta_1,\theta_2)u_0\,,\check{u}\big) + \big(\psi(\theta_0,\theta_2)u_1 + \psi(\theta_0,\theta_1)u_2\,,\,\check{u}\big) \\
&\quad + \big(\chi(\theta_0,\theta_1,\theta_2)q_0\,,\check{q}\big) + \big(\psi(\theta_0,\theta_2)q_1 + \psi(\theta_0,\theta_1)q_2\,,\,\check{q}\big)
\end{aligned}
\tag{4.20}
$$

for all $\check{\zeta} = (\check{u},\check{p},\check{q},\check{\eta}) \in Z$, where

$$
\chi(\theta_0,\theta_1,\theta_2) = \frac{\beta_2}{3\beta_0^2\kappa^4(\theta_0)}\big(u_1\cdot q_0 + u_0\cdot q_1\big) + \frac{\beta_1}{3\beta_0^2\kappa^4(\theta_0)}\big(u_2\cdot q_0 + u_0\cdot q_2\big)
$$

$$
\begin{aligned}
&- \frac{1}{3\beta_0\kappa^4(\theta_0)}\Big(u_1\cdot q_2 + u_2\cdot q_1 + \frac{2\beta_1\beta_2}{\beta_0^2}u_0\cdot q_0\Big)\\
&+ \frac{4}{9\beta_0^2\kappa^7(\theta_0)}\Big(u_1\cdot q_0 + u_0\cdot q_1 - \frac{\beta_1}{\beta_0}u_0\cdot q_0\Big)\Big(u_2\cdot q_0 + u_0\cdot q_2 - \frac{\beta_2}{\beta_0}u_0\cdot q_0\Big),
\end{aligned}
$$

and

$$
\psi(\theta_0,\theta_i) = \frac{1}{3\kappa^4(\theta_0)}\Big(\frac{\beta_i}{\beta_0^2}u_0\cdot q_0 - \frac{1}{\beta_0}u_i\cdot q_0 - \frac{1}{\beta_0}u_0\cdot q_i\Big),
$$

with $i = 1, 2$, and $\kappa(\theta)$ defined as in (4.14).

PROPOSITION 4.3. *The mapping* (4.15) $F : \Theta \to \Xi$, *where*

$$
\begin{aligned}
&\Theta \equiv Z \times \Lambda = [L^2_\rho\big(\Gamma; H(div, D)\big) \times L^2_\rho\big(\Gamma; L^2(D)\big)]^2 \times \mathbb{R}^+,\\
&\Xi = Z^* \quad (\textit{duality with respect to } (4.13)),
\end{aligned}
$$

is of class C^2 on $S \subset \Theta$

$$
S = [L^6_\rho\big(\Gamma; H(div, D) \cap L^6(D)\big) \times L^2_\rho\big(\Gamma; L^2(D)\big)]^2 \times \mathbb{R}^+,
$$

and $D^2_\theta F$ is bounded on all bounded sets of S.

Proof. Starting with (4.20), straightforward but somewhat tedious calculation shows that F is C^2 mapping on S, and $D^2_\theta F$ is bounded on bounded sets of S. $\square$

PROPOSITION 4.4. *Let $\theta_0 = \big((u_0, p_0, q_0, \eta_0), \beta_0\big)$ be a point in S such that $\xi_0 = F(\theta_0)$ is a regular value of F. Then,* $\ker D_{(u,p,q\eta)}F(\theta_0) = \{0\}$. *Moreover, if $\widetilde{Z} \subset Z$, then* $\ker D_{(u,p,q,\eta)}\widetilde{F}(\theta_0)|_{\widetilde{Z}} = \{0\}$ *as well.*

Proof. Without loss of generality [20], i.e., by redefining the origin in Θ, we may choose $\theta_0 = \big((u_\circ, p_\circ, q_\circ, \eta_\circ), \beta_\circ\big)$ with $\beta_\circ$ in an bounded interval of $\mathbb{R}^+$ and $u_\circ, q_\circ$ such that $\big(\frac{1}{\beta_\circ}u_\circ\cdot q_\circ\big)^{\frac{1}{3}} = \frac{1}{2}\big(\kappa_{\min} + \kappa_{\max}\big)$. Then from (4.2) and (4.14) we have that $\kappa(\theta_0) = \kappa_1$, and therefore (4.19) writes

$$
\begin{aligned}
\big(D_\zeta F(\theta_0)\theta_1, \check{\zeta}\big) = {}& \frac{1}{\kappa_1}\big(u_1\,, \check{u}\big) - \frac{1}{3\beta_\circ\kappa_1^4}\big((u_1\cdot q_\circ + u_\circ\cdot q_1)u_\circ\,, \check{u}\big) - (p_1, \nabla\cdot\check{u})\\
&+ (\nabla\cdot u_1, \check{p}) + \frac{1}{\kappa_1}\big(q_1\,, \check{q}\big) - \frac{1}{3\beta_\circ\kappa_1^4}\big((u_1\cdot q_\circ + u_\circ\cdot q_1)q_\circ\,, \check{q}\big)\\
&- (\eta_1, \nabla\cdot\check{q}) + (\mathbb{E}u_1, \check{q}) + (\nabla\cdot q_1 - \mathbb{E}p_1, \check{\eta}), \quad \forall\,\check{\zeta} = (\check{u}, \check{p}, \check{q}, \check{\eta}) \in Z
\end{aligned}
$$

from which it easily follows that $\ker D_{(u,p,q,\eta)}F(\theta_0) = \{0\}$. The result for the discrete operator $\widetilde{F}$ follows in the same manner. $\square$

We have now verified all the assumptions of Theorems 4.1 and 4.2 specialized to our setting. Then, the following error estimate follows directly from Theorem 4.2.

THEOREM 4.5. *Let $\big(u(\beta), p(\beta), q(\beta), \eta(\beta)\big) \in Z = [L^2_\rho\big(\Gamma; H(div, D)\big)\times L^2_\rho\big(\Gamma; L^2(D)\big)]^2$ and $\big(\widetilde{u}(\beta), \widetilde{p}(\beta), \widetilde{q}(\beta), \widetilde{\eta}(\beta)\big) \in \widetilde{Z} \subset Z$ denote solutions of the optimality system* (4.16) *and its discretization* (4.18). *Then,*

$$
\begin{aligned}
&\|u(\beta) - \widetilde{u}(\beta)\|_{L^2_\rho(\Gamma, H(div,D))} + \|p(\beta) - \widetilde{p}(\beta)\|_{L^2_\rho(\Gamma, L^2(D))}\\
&\qquad + \|q(\beta) - \widetilde{q}(\beta)\|_{L^2_\rho(\Gamma, H(div,D))} + \|\eta(\beta) - \widetilde{\eta}(\beta)\|_{L^2_\rho(\Gamma, L^2(D))}\\
&\le C \inf_{(\check{u},\check{p},\check{q},\check{\eta})\in\widetilde{Z}} \Big\{\|u(\beta) - \check{u}(\beta)\|_{L^2_\rho(\Gamma, H(div,D))} + \|p(\beta) - \check{p}(\beta)\|_{L^2_\rho(\Gamma, L^2(D))}\\
&\qquad\qquad + \|q(\beta) - \check{q}(\beta)\|_{L^2_\rho(\Gamma, H(div,D))} + \|\eta(\beta) - \check{\eta}(\beta)\|_{L^2_\rho(\Gamma, L^2(D))}\Big\}.
\end{aligned} \tag{4.21}
$$

5. Numerical Experiments.

5.1. Sensitivity Analysis for the Parameter Estimation. Consider the state equations:

$$\left\{\begin{aligned} u &= -\kappa\nabla p && \text{in } \Omega\times D,\\ \nabla\cdot u &= f && \text{in } \Omega\times D,\\ p &= 0 && \text{on } \Omega\times\partial D. \end{aligned}\right. \tag{5.1}$$

We introduce the adjoint equations:

$$\left\{\begin{aligned} \nabla\cdot q &= p-\overline{p} && \text{in } \Omega\times D,\\ \kappa^{-1}q+\nabla\eta &= -(u-\overline{u}) && \text{in } \Omega\times D,\\ \eta &= 0 && \text{on } \Omega\times\partial D. \end{aligned}\right. \tag{5.2}$$

Define the cost functional:

$$\begin{aligned} J_3(Y_i, i=1\dots N) = {}& \frac{1}{2}\mathbb{E}\left[\|u(Y_i,\cdot)-\overline{u}(Y_i,\cdot)\|^2_{L^2(D)}\right] + \frac{1}{2}\mathbb{E}\left[\|p(Y_i,\cdot)-\overline{p}(Y_i,\cdot)\|^2_{L^2(D)}\right]\\ &+\frac{\beta}{2}\mathbb{E}\left[\|\kappa(Y_i,\cdot)\|^2_{L^2(D)}\right]. \end{aligned}$$

We assume that the map $\kappa\to u$ is differentiable. Then, the sensitivity equations are the following:

$$\left\{\begin{aligned} u+\epsilon u_s &= -\left((\kappa+\epsilon\kappa_s)\nabla(p+\epsilon p_s)\right) && \text{in } \Omega\times D,\\ \nabla\cdot(u+\epsilon u_s) &= f && \text{in } \Omega\times D,\\ p_s &= 0 && \text{on } \Omega\times\partial D. \end{aligned}\right. \tag{5.3}$$

Multiplying out and using the state equations (5.1), we get:

$$\left\{\begin{aligned} \kappa^{-1}u_s-\kappa^{-2}\kappa_s u &= -\nabla p_s && \text{in } \Omega\times D,\\ \nabla\cdot u_s &= 0 && \text{in } \Omega\times D,\\ p_s &= 0 && \text{on } \Omega\times\partial D. \end{aligned}\right. \tag{5.4}$$

Next, we multiply by the adjoint variables q, η and integrate over D:

$$\left\{\begin{aligned} \textstyle\int_D \kappa^{-1}u_s q - \int_D \kappa^{-2}\kappa_s u q &= -\textstyle\int_D \nabla p_s q && \text{in } \Omega\times D,\\ \textstyle\int_D \nabla\cdot u_s\eta &= 0 && \text{in } \Omega\times D,\\ p_s &= 0 && \text{on } \Omega\times\partial D. \end{aligned}\right. \tag{5.5}$$

Integrate by parts:

$$\left\{\begin{aligned} \textstyle\int_D \kappa^{-1}u_s q - \int_D \kappa^{-2}\kappa_s u q &= \textstyle\int_D p_s\nabla\cdot q && \text{in } \Omega\times D,\\ \textstyle\int_D u_s\nabla\eta &= 0 && \text{in } \Omega\times D,\\ p_s &= 0 && \text{on } \Omega\times\partial D. \end{aligned}\right. \tag{5.6}$$

There are no boundary terms since $\eta=0$ and $p_s=0$ on $\Omega\times\partial D$.
Take expectation and add the first two identities of (5.6):

$$\mathbb{E}\left[\int_D u_s\left(\kappa^{-1}q+\nabla\eta\right)\right]-\mathbb{E}\left[\int_D \kappa^{-2}\kappa_s u q\right]=\mathbb{E}\left[\int_D p_s\nabla\cdot q\right].$$

Using the adjoint equations (5.2), the previous identity becomes:

$$-\mathbb{E}\left[\int_D u_s(u-\overline{u})\right] - \mathbb{E}\left[\int_D \kappa^{-2}\kappa_s uq\right] = \mathbb{E}\left[\int_D p_s(p-\overline{p})\right].$$

We give below the pseudocode for the **Adjoint variable-based Algorithm**, as in [25]:

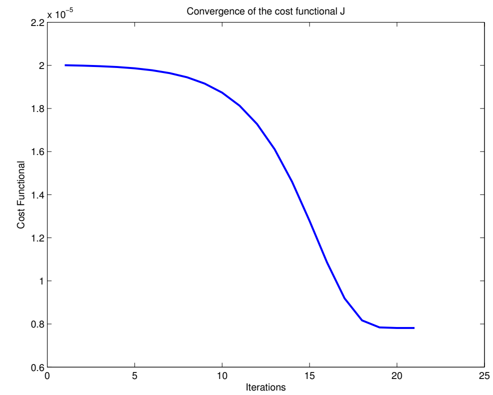
INITIALIZATION: $i \leftarrow 1$, $RelError \leftarrow 1000$, Choose initial conditions for Y, $\epsilon = 1$, $\epsilon \leftarrow 2\epsilon/3$
while $RelError > tol$ **do**
  $\epsilon \leftarrow 3\epsilon/2$, $i \leftarrow i+1$
  Solve Adjoint Equations (for the adjoint variables)
  Solve Standard Gradient Update, i.e. $Y_{i+1} = Y_i - \epsilon\frac{dJ}{dY_i}$
  Solve State Equations
  Evaluate $J_n(i)$
  while $J_n^{(i)} > J_n^{(i-1)}$ **do**
    $\epsilon \leftarrow \epsilon/10$
    Solve Standard Gradient Update
    Solve State Equations
    Evaluate $J_n(i)$
  end while
  $RelError \leftarrow \left|J_n^{(i)} - J_n^{(i-1)}\right| / \left|J_n^{(i)}\right|$
end while

The numerical experiments were performed using MATLAB R2012a and were solved on a square domain $[0,1]\times[0,1]$. The convergence is computed on a 40×40 spatial mesh, with Dirichlet boundary conditions. For solving the equation (5.1) numerically, an upwind scheme is used to find the effective diffusion coefficient and central difference for finding the hydraulic gradient. We assume the true random diffusion coefficient $\overline{\kappa}$ and the exact solution $\overline{p}$ to be given by:

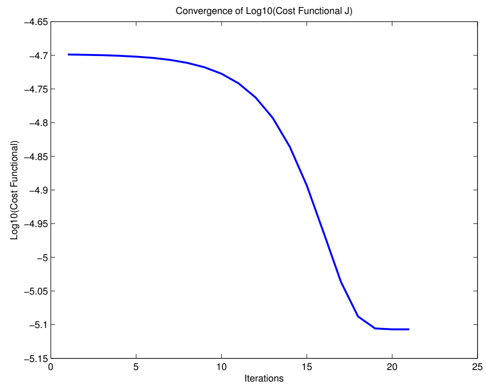
$$\overline{\kappa}(\omega,x) = (1+x^2+y^2) + \frac{1}{N} * \sum_{n=1}^{N} \cos(n\pi x)\cdot\cos(n\pi y)\, Y_n(\omega)$$
$$\overline{p}(\omega,x) = \sum_{n=1}^{N} \sin(n\pi x)\cdot\sin(n\pi y)\, Y_n(\omega)$$

and then we calculate the source $f(\omega,x)$.
To understand the dynamics that the computational model produces, we first present some sample simulations. The tolerance was taken 10^{-4}, the step size for the adjoint algorithm is $\epsilon = 1$ and the coefficient $\beta = 10^{-6}$ in the cost functional formula.

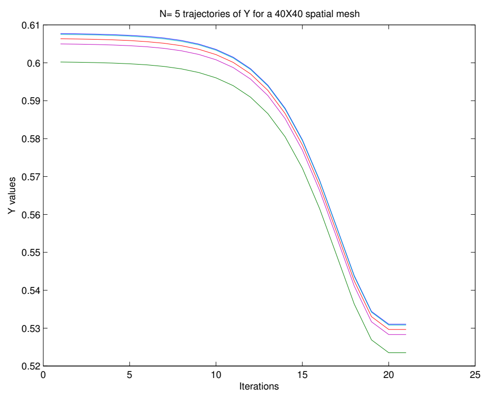
5.2. Numerical Experiments for the Deterministic Elliptic Case. The exact values used for producing simulated measurements were 0.5 for the $Y_i, i = 1,\ldots,N$ in the formula for the diffusion coefficient k. Figures 5.1(a) and 5.1(b) show the plot for the cost functional J and the logarithm of J to base 10. The trajectories of the $N=5$ Ys, the cross-section of target solution versus estimated solution, the cross-section of target diffusion versus estimated diffusion are presented in figures 5.1(c), (d) and (e) respectively.



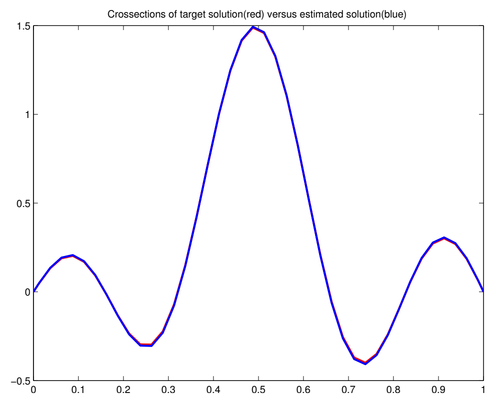
(a)



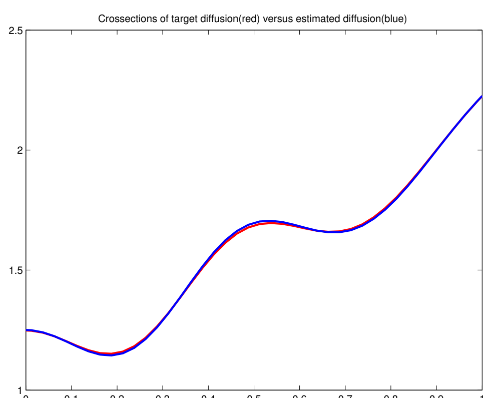
(b)



(c)



(d)



(e)

Fig. 5.1: **Deterministic Case:** (a)Cost functional J, (b)Log10(J), (c)N=5 trajectories of Y's, (d)cross-section of target solution versus estimated solution, (e)cross-sections of target diffusion versus estimated diffusion for a 40x40 grid; tol=10^{-4}, $\epsilon = 1$, $\beta = 10^{-6}$. The exact values of Ys are 0.5.

5.3. Numerical Experiments for the Stochastic Elliptic Case. The exact values used for producing simulated measurements were considered uniformly distributed random numbers for the $Y_i, i = 1, \ldots, n$. To understand the dynamics that the computational model produces, we present sample simulations on a 40x40 spatial mesh, where we first considered 10 realizations (see Figures 5.2 and 5.3) and then 50 realizations (see Figures 5.4 and 5.5) for our stochastic model.

In the **first simulation** where only 10 realizations were considered, we observed that for achieving the same tolerance of 10^{-4} for the relative error, the cost functional J_3 only needs to do 27 iterations, whereas J_4 and J_5 require 76 and 61 iterations respectively. By plotting cross-sections, we observed that our estimated solutions corresponding to either J_3, J_4 or J_5 approximate very well the mean of the target solution, while the variance of the target solution is better approximated when using the solution corresponding to J_3 cost functional. When considering the cross-sections for the diffusion coefficient, the mean and variance of our estimated diffusion were not doing so well in approximating the mean and variance of the target diffusion coefficient. One explanation would be the fact that our cost functionals try to minimize the difference between the estimated and target solutions, whereas the difference between the estimated and target diffusion coefficient is never taken into account in the formulas of the cost functionals.

In the **second simulation** with 50 realizations being considered, the cost functional J_3 only needs 22 iterations, whereas J_4 and J_5 require 47 and 95 iterations respectively. By plotting cross-sections, again it was observed that our estimated solutions corresponding to either J_3, J_4 or J_5 approximate really well the mean of the target solution, whereas for the variance, it seems the solution corresponding to J_5 is closer to the variance of the target solution. By looking at the cross-sections for the diffusion coefficient, we can see the mean and variance of our estimated diffusion are not doing so great in approximating the mean and variance of the target diffusion coefficient.

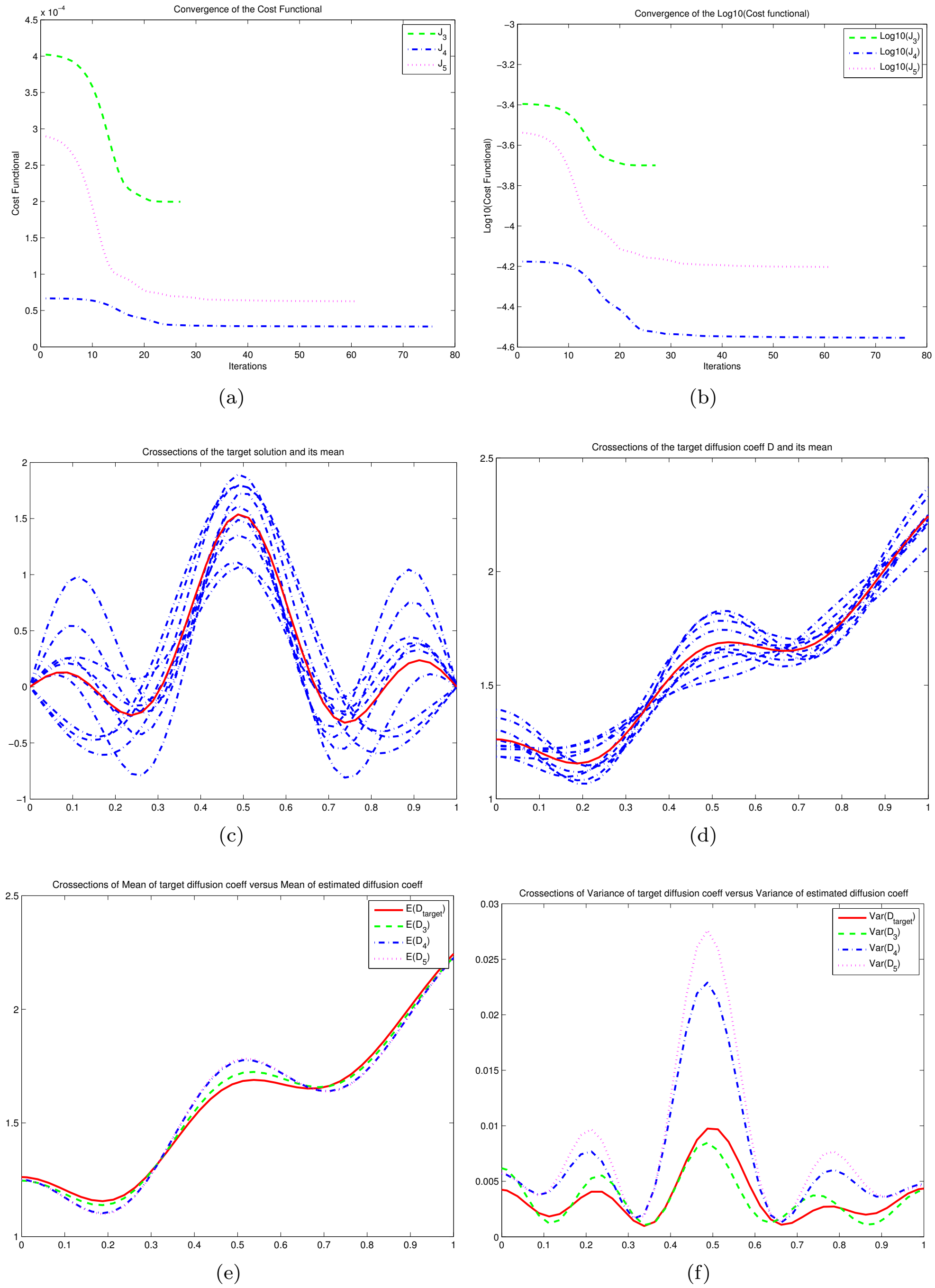


Fig. 5.2: (a)J, (b)$Log_{10}(J)$ and cross-sections for: (c)target solution, (d)target diffusion, (e)mean of target diffusion vs. mean of estimated diffusion, (f)variance of target diffusion vs. variance of estimated diffusion. Grid considered is 40x40, tol=10^{-4}, $\epsilon = 1, \beta = 10^{-6}$, runs=10. The target values of N=5 Ys are random.

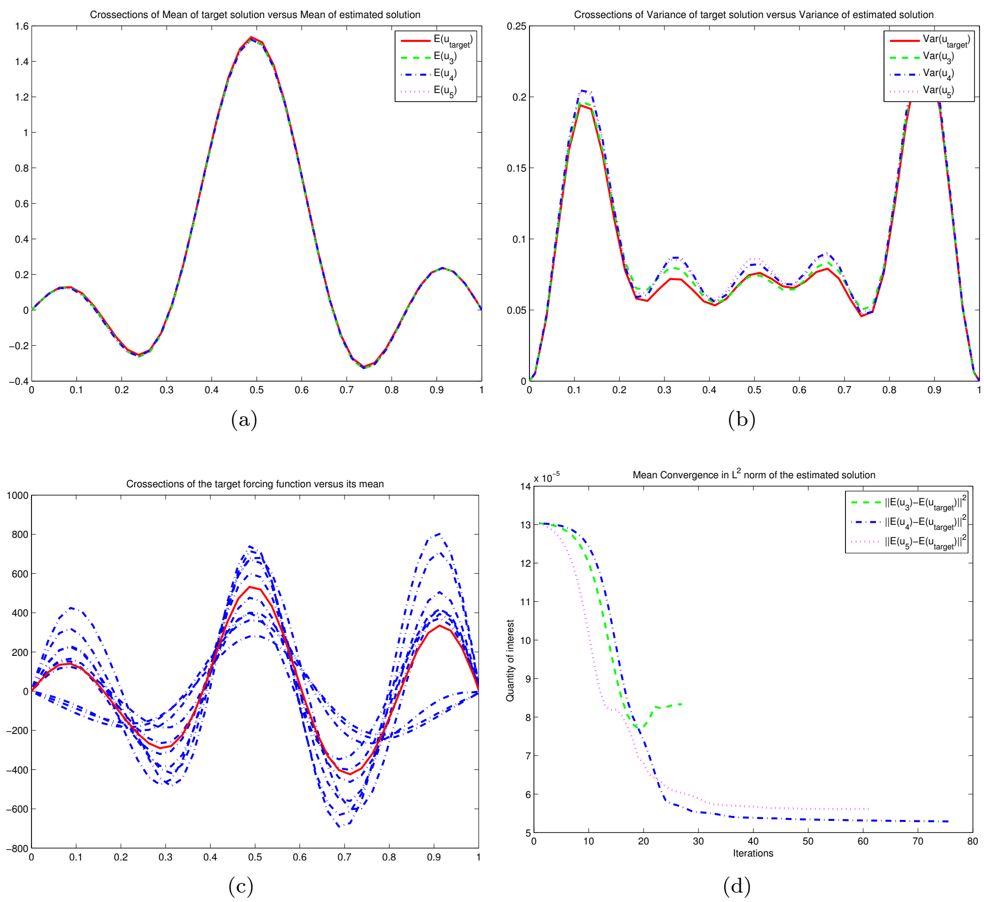
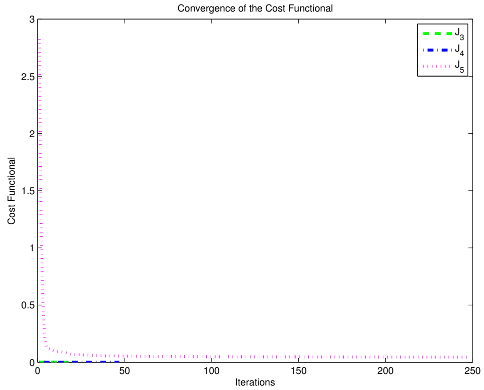
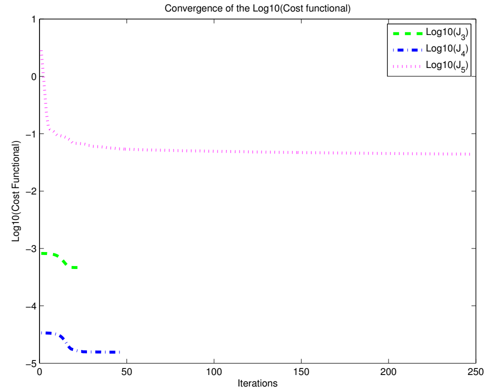


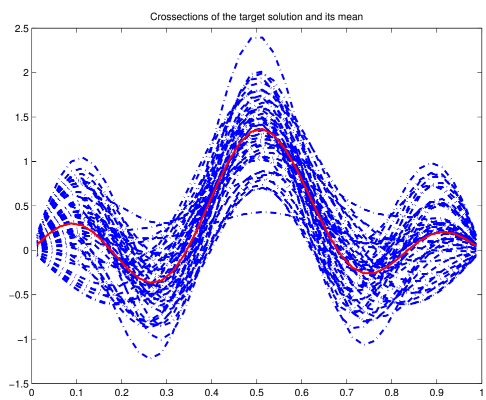
Fig. 5.3: cross-sections for: (a)mean of target solution vs. mean of estimated solution, (b)variance of target solution vs. variance of estimated solution, (c)forcing function f, (d)mean convergence in L^2 norm of estimated solution. Grid considered is 40x40, tol=10^{-4}, $\epsilon = 1, \beta = 10^{-6}$, runs=10. The target values of N=5 Ys are random.



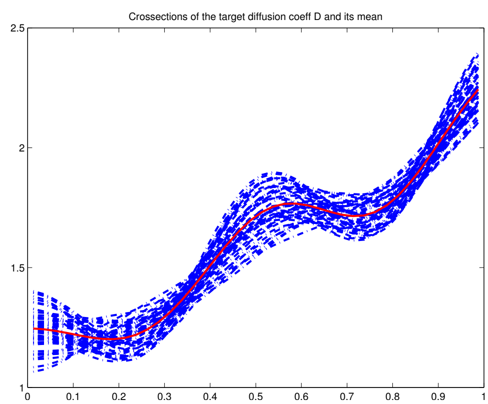
(a)



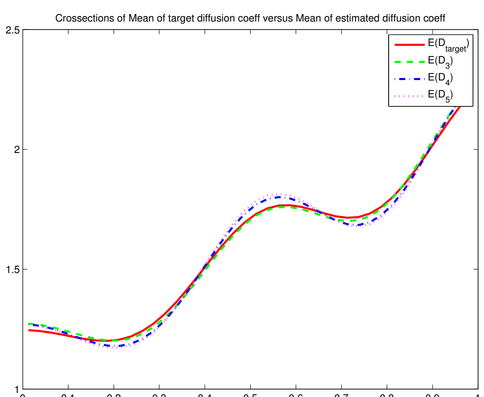
(b)



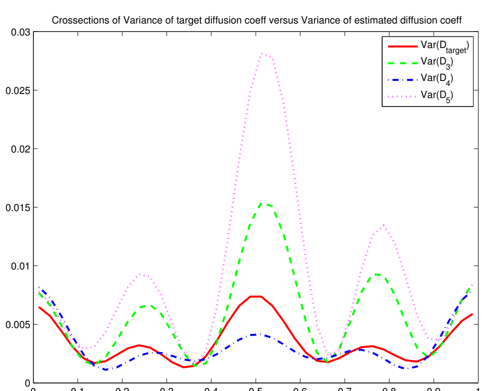
(c)



(d)

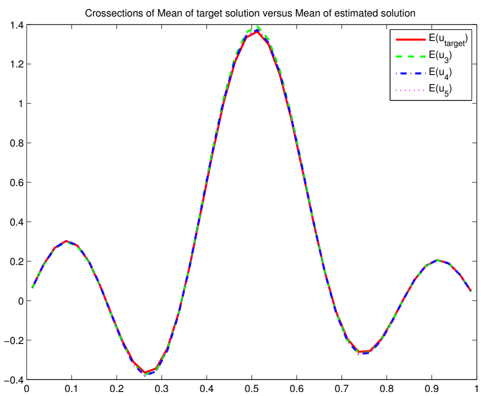


(e)

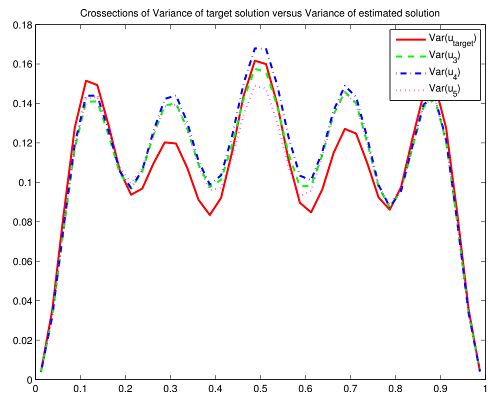


(f)

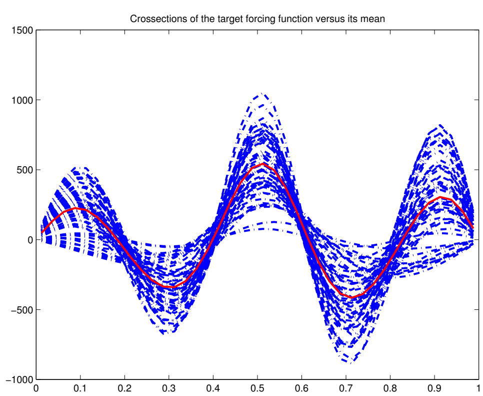
Fig. 5.4: (a)J, (b)$Log_{10}(J)$ and cross-sections for:(c)target solution, (d)target diffusion, (e)mean of target diffusion vs. mean of estimated diffusion, (f)variance of target diffusion vs. variance of estimated diffusion. Grid considered is 40x40, tol=10^{-4}, $\epsilon = 1, \beta = 10^{-6}$, runs=50. The target values of N=5 Ys are random.



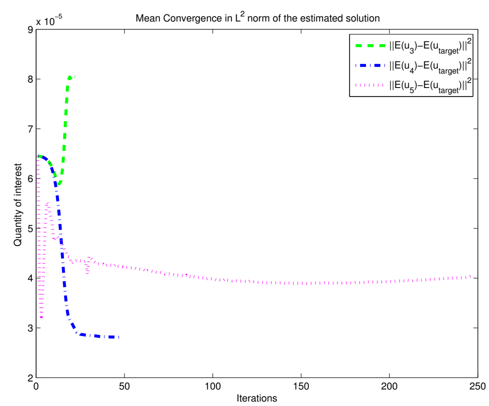
(a)



(b)



(c)



(d)

Fig. 5.5: cross-sections for: (a)mean of target solution vs. mean of estimated solution, (b)variance of target solution vs. variance of estimated solution, (c)forcing function f, (d)mean convergence in L^2 norm of estimated solution. Grid considered is 40x40, tol=10^{-4}, $\epsilon = 1, \beta = 10^{-6}$, runs=50. The target values of N=5 Ys are random.

REFERENCES

[1] R.J. Adler, *The geometry of random fields*, John Wiley & Sons, Chichester, 1981.

[2] Alen Alexanderian, Noemi Petra, Georg Stadler, and Omar Ghattas, *A-optimal design of experiments for infinite-dimensional Bayesian linear inverse problems with regularized ℓ_0-sparsification*, SIAM J. Sci. Comput. **36** (2014), no. 5, A2122–A2148. MR 3257642

[3] Douglas Allaire and Karen Willcox, *A mathematical and computational framework for multifidelity design and analysis with computer models*, Int. J. Uncertain. Quantif. **4** (2014), no. 1, 1–20. MR 3249674

[4] Ivo Babuška, Fabio Nobile, and Raúl Tempone, *A stochastic collocation method for elliptic partial differential equations with random input data*, SIAM J. Numer. Anal. **45** (2007), no. 3, 1005–1034 (electronic). MR MR2318799 (2008e:65372)

[5] Velamur Asokan Badri Narayanan and Nicholas Zabaras, *Stochastic inverse heat conduction using a spectral approach*, Internat. J. Numer. Methods Engrg. **60** (2004), no. 9, 1569–1593. MR 2068927

[6] Guillaume Bal, Ian Langmore, and Youssef Marzouk, *Bayesian inverse problems with Monte Carlo forward models*, Inverse Probl. Imaging **7** (2013), no. 1, 81–105. MR 3031839

[7] H.T. Banks and K. Kunisch, *Estimation techniques for distributed parameter systems*, Birkhäuser, Boston, 1989.

[8] V. Barbu and K. Kunisch, *Identification of nonlinear elliptic equations*, Appl. Math. Optim **33** (1996), 139–167.

[9] Viorel Barbu, *Analysis and control of nonlinear infinite-dimensional systems*, Mathematics in Science and Engineering, vol. 190, Academic Press Inc., Boston, MA, 1993. MR MR1195128 (93j:49002)

[10] I. Bilionis and N. Zabaras, *Solution of inverse problems with limited forward solver evaluations: a Bayesian perspective*, Inverse Problems **30** (2014), no. 1, 015004, 32. MR 3151683

[11] Tan Bui-Thanh and Omar Ghattas, *An analysis of infinite dimensional Bayesian inverse shape acoustic scattering and its numerical approximation*, SIAM/ASA J. Uncertain. Quantif. **2** (2014), no. 1, 203–222. MR 3283906

[12] Peng Chen and Alfio Quarteroni, *Weighted reduced basis method for stochastic optimal control problems with elliptic PDE constraint*, SIAM/ASA J. Uncertain. Quantif. **2** (2014), no. 1, 364–396. MR 3283913

[13] Peng Chen, Alfio Quarteroni, and Gianluigi Rozza, *Stochastic optimal Robin boundary control problems of advection-dominated elliptic equations*, SIAM J. Numer. Anal. **51** (2013), no. 5, 2700–2722. MR 3115461

[14] Peng Chen, Nicholas Zabaras, and Ilias Bilionis, *Uncertainty propagation using infinite mixture of Gaussian processes and variational Bayesian inference*, J. Comput. Phys. **284** (2015), 291–333. MR 3303621

[15] G. Christakos, *Random field models in earth sciences*, Academic Press, New York, NY, 1992.

[16] Francis Clarke, *Functional analysis, calculus of variations and optimal control*, Graduate Texts in Mathematics, vol. 264, Springer, London, 2013. MR 3026831

[17] Tiangang Cui, Youssef M. Marzouk, and Karen E. Willcox, *Data-driven model reduction for the Bayesian solution of inverse problems*, Internat. J. Numer. Methods Engrg. **102** (2015), no. 5, 966–990. MR 3341244

[18] M. Dentz, D.M. Tartakovsky, E. Abarca, A. Guadagnini, X. Sanchez-Vila, and J. Carrera, *Variable-density flow in porous media*, J. Fluid Mech. **561** (2006), 209–235.

[19] James P. Fink and Werner C. Rheinboldt, *On the discretization error of parametrized nonlinear equations*, SIAM J. Numer. Anal. **20** (1983), no. 4, 732–746. MR 708454 (85i:65072)

[20] ———, *Solution manifolds and submanifolds of parametrized equations and their discretization errors*, Numer. Math. **45** (1984), no. 3, 323–343. MR 769244 (86d:58018)

[21] ———, *Local error estimates for parametrized nonlinear equations*, SIAM J. Numer. Anal. **22** (1985), no. 4, 729–735. MR 795950 (87a:65106)

[22] ———, *A geometric framework for the numerical study of singular points*, SIAM J. Numer. Anal. **24** (1987), no. 3, 618–633. MR 888753 (88h:65118)

[23] M. Frangos, Y. Marzouk, K. Willcox, and B. van Bloemen Waanders, *Surrogate and reduced-order modeling: a comparison of approaches for large-scale statistical inverse problems*, Large-scale inverse problems and quantification of uncertainty, Wiley Ser. Comput. Stat., Wiley, Chichester, 2011, pp. 123–149. MR 2856654

[24] D. Galbally, K. Fidkowski, K. Willcox, and O. Ghattas, *Non-linear model reduction for uncertainty quantification in large-scale inverse problems*, Internat. J. Numer. Methods Engrg. **81** (2010), no. 12, 1581–1608. MR 2642821 (2011a:65024)

[25] M.R. Garvie, P. Maini, and C. Trenchea, *An efficient and robust numerical algorithm for*

estimating parameters in Turing systems, J. Comp. Phys. **229** (2010), no. 19, 7058–7071.

[26] R. Ghanem and S. Dham, *Stochastic finite element analysis for multiphase flow in heterogeneous porous media*, Transport in Porous Media **32** (1998), 239–262.

[27] V. Girault and P.-A. Raviart, *Finite element methods for Navier-Stokes equations*, Springer Series in Computational Mathematics, vol. 5, Springer-Verlag, Berlin, 1986, Theory and algorithms.

[28] M. Gunzburger, C. Trenchea, and C. Webster, *Error estimates for a stochastic collocation approach to identification and control problems for random elliptic pdes*, Tech. report, University of Pittsburgh, 2015.

[29] Max D. Gunzburger, Hyung-Chun Lee, and Jangwoon Lee, *Error estimates of stochastic optimal Neumann boundary control problems*, SIAM J. Numer. Anal. **49** (2011), no. 4, 1532–1552.

[30] V. Isakov, *Inverse problems for partial differential equations*, second ed., Applied Mathematical Sciences, vol. 127, Springer, New York, 2006. MR MR2193218 (2006h:35279)

[31] Kazufumi Ito and Karl Kunisch, *Lagrange multiplier approach to variational problems and applications*, Advances in Design and Control, vol. 15, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2008. MR MR2441683 (2009g:49001)

[32] A. Keese and Matthies H.G., *Parallel computation of stochastic groundwater flow*, NIC Symposium 2004, Proceedings **20** (2003), 399–408.

[33] D. P. Kouri, M. Heinkenschloss, D. Ridzal, and B. G. van Bloemen Waanders, *A trust-region algorithm with adaptive stochastic collocation for PDE optimization under uncertainty*, SIAM J. Sci. Comput. **35** (2013), no. 4, A1847–A1879. MR 3073358

[34] ———, *Inexact objective function evaluations in a trust-region algorithm for PDE-constrained optimization under uncertainty*, SIAM J. Sci. Comput. **36** (2014), no. 6, A3011–A3029. MR 3293456

[35] Jesper Kristensen and Nicholas J. Zabaras, *Bayesian uncertainty quantification in the evaluation of alloy properties with the cluster expansion method*, Comput. Phys. Commun. **185** (2014), no. 11, 2885–2892. MR 3252545

[36] O. Ladyzhenskaya and N. Ural'tseva, *Equations à dérivées partielles de type elliptique*, Dunod, Paris, 1968.

[37] Toni Lassila, Andrea Manzoni, Alfio Quarteroni, and Gianluigi Rozza, *A reduced computational and geometrical framework for inverse problems in hemodynamics*, Int. J. Numer. Methods Biomed. Eng. **29** (2013), no. 7, 741–776. MR 3081353

[38] Yu.N. Lazarev, P.V. Petrov, and D.M. Tartakovsky, *Interface dynamics in randomly heterogeneous porous media*, Advances in Water Resources **28** (2005), 393–403.

[39] Chad Lieberman, Karen Willcox, and Omar Ghattas, *Parameter and state model reduction for large-scale statistical inverse problems*, SIAM J. Sci. Comput. **32** (2010), no. 5, 2523–2542. MR 2684726 (2011h:65007)

[40] J.-L. Lions, *Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles*, Avant propos de P. Lelong, Dunod, Paris, 1968. MR MR0244606 (39 #5920)

[41] ———, *Some methods in the mathematical analysis of systems and their control*, Kexue Chubanshe (Science Press), Beijing, 1981. MR MR664760 (84m:49003)

[42] Xiang Ma and Nicholas Zabaras, *An efficient Bayesian inference approach to inverse problems based on an adaptive sparse grid collocation method*, Inverse Problems **25** (2009), no. 3, 035013, 27. MR 2480183 (2010e:65082)

[43] James Martin, Lucas C. Wilcox, Carsten Burstedde, and Omar Ghattas, *A stochastic Newton MCMC method for large-scale statistical inverse problems with application to seismic inversion*, SIAM J. Sci. Comput. **34** (2012), no. 3, A1460–A1487. MR 2970260

[44] Youssef Marzouk and Dongbin Xiu, *A stochastic collocation approach to Bayesian inference in inverse problems*, Commun. Comput. Phys. **6** (2009), no. 4, 826–847. MR 2672325 (2011g:62072)

[45] Youssef M. Marzouk and Habib N. Najm, *Dimensionality reduction and polynomial chaos acceleration of Bayesian inference in inverse problems*, J. Comput. Phys. **228** (2009), no. 6, 1862–1902. MR 2500666 (2010b:65023)

[46] Youssef M. Marzouk, Habib N. Najm, and Larry A. Rahn, *Stochastic spectral methods for efficient Bayesian solution of inverse problems*, J. Comput. Phys. **224** (2007), no. 2, 560–586. MR 2330284 (2008b:65074)

[47] H. N. Najm, B. J. Debusschere, Y. M. Marzouk, S. Widmer, and O. P. Le Maître, *Uncertainty quantification in chemical systems*, Internat. J. Numer. Methods Engrg. **80** (2009), no. 6-7, 789–814. MR 2583862

[48] Pekka Neittaanmaki, Jürgen Sprekels, and Dan Tiba, *Optimization of elliptic systems*, Springer Monographs in Mathematics, Springer, New York, 2006, Theory and applications. MR

2183776 (2006j:49003)

[49] F. Nobile, R. Tempone, and C. G. Webster, *An anisotropic sparse grid stochastic collocation method for partial differential equations with random input data*, SIAM J. Numer. Anal. **46** (2008), no. 5, 2411–2442. MR MR2421041 (2009c:65331)

[50] H. Osnes and H.P. Langtangen, *An efficient probabilistic finite element method for stochastic groundwater flow*, Advances in Water Resources **22** (1998), 185–195.

[51] J. Ray, Y. M. Marzouk, and H. N. Najm, *A Bayesian approach for estimating bioterror attacks from patient data*, Stat. Med. **30** (2011), no. 2, 101–126. MR 2758268

[52] D.M. Tartakovsky, *Probabilistic risk analysis in subsurface hydrology*, Geophys. Res. Lett. **34** (2007), L05404.

[53] D.M. Tartakovsky, A. Guadagnini, and M. Riva, *Stochastic averaging of nonlinear flows in heterogeneous porous media*, J. Fluid Mech. **492** (2003), 47–62.

[54] D.M. Tartakovsky and S.P. Neuman, *Extension of "transient flow in bounded randomly heterogeneous domains 1. Exact conditional moment equations and recursive approximations"*, Water Resour. Res. **35** (1999), no. 6, 1921–1925.

[55] E. Vanmarcke, *Random Fields: Analysis and Synthesis*, 3^{rd} ed., The MIT Press, Cambridge, MA, 1988.

[56] Jiang Wan and Nicholas Zabaras, *A Bayesian approach to multiscale inverse problems using the sequential Monte Carlo method*, Inverse Problems **27** (2011), no. 10, 105004, 25. MR 2835979 (2012h:62365)

[57] Jingbo Wang and Nicholas Zabaras, *Hierarchical Bayesian models for inverse problems in heat conduction*, Inverse Problems **21** (2005), no. 1, 183–206. MR 2146171 (2006a:80005)

[58] D. Xiu and D.M. Tartakovsky, *Numerical methods for differential equations in random domains*, SIAM J. Sci. Comput. **28** (2006), no. 3, 1167–1185.

[59] N. Zabaras, *Solving stochastic inverse problems: a sparse grid collocation approach*, Large-scale inverse problems and quantification of uncertainty, Wiley Ser. Comput. Stat., Wiley, Chichester, 2011, pp. 291–319. MR 2856661

[60] N. Zabaras and B. Ganapathysubramanian, *A scalable framework for the solution of stochastic inverse problems using a sparse grid collocation approach*, J. Comput. Phys. **227** (2008), no. 9, 4697–4735. MR 2406554 (2009i:65013)