EXACT SOLUTIONS OF THE STOCHASTIC NAVIER-STOKES EQUATIONS

AZIZ TAKHIROV *

Abstract. This report develops a class of exact solutions based on Green-Taylor's vortex [5], [6] to stochastic Navier-Stokes equations to test the accuracy of the various models and methods.

Key Words: Stochastic Navier-Stokes, Green-Taylor vortex.

Mathematics Subject Classification:

1. Introduction. In Computational Fluid Dynamics (CFD), all the approaches to Uncertainty Quantification (UQ) seek to quantify the uncertainty in approximations to the expectation, moments and other statistical quantities of interest in a flow. Validation of various approaches (e.g., polynomial chaos [7], stochastic collocation [8]) to UQ in CFD requires as a first step exact solutions of the stochastic NSE with with stochasticity introduced in various places. In this report, we present exact solutions of the stochastic Navier-Stokes equations extending the Green-Taylor vortex solutions [1] to include stochastic forcing, initial conditions and viscosity. Green-Taylor vortex solutions have been extensively used in CFD, see e.g. [4], [3].

For a random variable ξ, we therefore consider the stochastic Navier-Stokes equations with stochasticity introduced in

(i) the body force by a Brownian motion $\mathbf{f} = \sigma \dot{W}(t)$ with $\sigma = const$, or
(ii) the initial condition $\mathbf{u}(x, 0; \xi) = \mathbf{a}(x)g(\xi)$, or
(iii) the viscosity $\nu(\xi)$.

Let $\mathbf{a}(x)$ be a Stokes eigenfunction

$$\triangle \mathbf{a} = \lambda \mathbf{a},\ \nabla \cdot \mathbf{a} = 0 \text{ in } \Omega \text{ and } \mathbf{a}(x) = \mathbf{0} \text{ on } \partial\Omega. \tag{1.1}$$

Let $\mathbf{u}(x, t; \xi),\ x \in \Omega \subset R^2$ satisfy

$$\begin{cases} \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} - \nu \triangle \mathbf{u} + \nabla p = \mathbf{f}, \\ \nabla \cdot \mathbf{u} = 0, \\ \mathbf{u} = \mathbf{0} \text{ on } \partial\Omega, \\ \mathbf{u}(x, 0; \xi) = \mathbf{a}(x)g(\xi). \end{cases} \tag{1.2}$$

We prove the following:

THEOREM 1.1.

(i) If $\mathbf{f} = \sigma \dot{W}(t)$ with $\sigma = const, \nu = const$ and $g \equiv 1$, then the exact solution of (1.2) *is*

$$\mathbf{u}(x, t) = \mathbf{a}\left(x - \int_0^t \sigma W(s) ds\right) e^{\lambda \nu t} + \sigma W(t), \tag{1.3}$$

$$\nabla p(x, t) = -\left(\mathbf{u} - \sigma W(t)\right) \cdot \nabla \mathbf{u}. \tag{1.4}$$

(ii) If $\mathbf{f} = \mathbf{0}, \nu = const$ and $g = g(\xi)$, then the exact solution of (1.2) *is*

$$\mathbf{u}(x, t; \xi) = e^{\lambda \nu t} \mathbf{a}(x) g(\xi), \tag{1.5}$$

$$\nabla p(x, t; \xi) = -\mathbf{u}(x, t; \xi) \cdot \nabla \mathbf{u}(x, t; \xi). \tag{1.6}$$

(iii) If $\mathbf{f} = \mathbf{0}, \nu = \nu(\xi)$ and $g \equiv 1$, then the exact solution of (1.2) *is*

$$\mathbf{u}(x, t; \xi) = e^{\lambda \nu(\xi) t} \mathbf{a}(x), \tag{1.7}$$

$$\nabla p(x, t; \xi) = -\mathbf{u}(x, t; \xi) \cdot \nabla \mathbf{u}(x, t; \xi). \tag{1.8}$$

These are the stochastic extensions of the famous Green-Taylor vortices.

*Department of Mathematics, University of Pittsburgh, Pittsburgh, PA, 15260, USA. email: azt7@pitt.edu.

2. The proof. First, we will briefly recall the construction of the Green-Taylor solution to deterministic Navier-Stokes equations

$$\mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v} = \nu\triangle\mathbf{v} - \nabla q,\ \nabla\cdot\mathbf{v} = 0 \text{ and } \mathbf{v}(x,0) = \mathbf{a}(x). \tag{2.1}$$

LEMMA 2.1. *(from [1]). Suppose* $\mathbf{a}(x)$ *satisfies* (1.1). *Then* $\mathbf{v} = e^{\nu\lambda t}\mathbf{a}$ *satisfies* $\nabla\times(\mathbf{v}\cdot\nabla\mathbf{v}) = \mathbf{0}$ *and* (2.1) *with pressure* q *such that* $\nabla q = -\mathbf{v}\cdot\nabla\mathbf{v}$.

Proof. Since $\nabla\cdot\mathbf{a} = 0$, then $\nabla\cdot\mathbf{u} = 0$. Futher, $\mathbf{u}_t = \nu\lambda\mathbf{u} = \nu\triangle\mathbf{u}$ and hence it only remains to prove that nonlinear term is a gradient. This is equivalent to showing

$$\frac{\partial}{\partial y}\left(u_1\frac{\partial u_1}{\partial x} + u_2\frac{\partial u_1}{\partial y}\right) = \frac{\partial}{\partial x}\left(u_1\frac{\partial u_2}{\partial x} + u_2\frac{\partial u_2}{\partial y}\right), \tag{2.2}$$

which directly follows from (1.1). □

$$\begin{aligned} u_1(x,t) &= -\cos(n\pi x)\sin(n\pi y)e^{-2\pi^2n^2\nu t},\\ u_2(x,t) &= \sin(n\pi x)\cos(n\pi y)e^{-2\pi^2n^2\nu t},\\ p(x,t) &= -\frac{1}{4}\left(\cos(2n\pi x) + \cos(2n\pi y)\right)e^{-4\pi^2n^2\nu t}, \end{aligned}$$

which corresponds to $\lambda = -2\pi^2n^2$, where n is any positive integer. Different linear combinations of eigenfunctions corresponding to λ gives rise to complex flow patterns [1].

Proof of the Theorem 1.1. From [1], it follows that if $\mathbf{a}(x)$ is a Stokes eigenfunction and since, for the last two cases in Theorem 1.1, $\mathbf{u}(x,t;\xi)$ takes the form $\mathbf{u} = \mathbf{a}(x)G(t,\xi)$, we have that $\nabla\times(\mathbf{u}\cdot\nabla\mathbf{u}) \equiv \mathbf{0}$. Then, as in [1], nonlinearity is balanced by the pressure $\nabla p = -\mathbf{u}\cdot\nabla\mathbf{u}$.

(i) Note that by using Ito's formula [2], the solution of (2.1), $(\mathbf{v}, q)$ is related to that of (1.2) by

$$\begin{cases} \mathbf{u}(x,t) = \mathbf{v}\left(x - \int\limits_0^t \sigma W(s)ds, t\right) + \sigma W(t),\\ p(x,t) = q\left(x - \int\limits_0^t \sigma W(s)ds, t\right). \end{cases} \tag{2.3}$$

Thus, by Lemma 2.1 we have that

$$\mathbf{u}(x,t) = \mathbf{a}\left(x - \int_0^t \sigma W(s)ds\right)e^{\lambda\nu t} + \sigma W(t), \tag{2.4}$$

along with $\mathbf{u}(x,0) = \mathbf{a}(x)$. Since $\nabla q = -\mathbf{v}\cdot\nabla\mathbf{v} = (\sigma W(t) - \mathbf{u})\cdot\nabla\mathbf{u}$, we obtain that

$$\nabla\times(\sigma W(t) - \mathbf{u})\cdot\nabla\mathbf{u} = \mathbf{0}.$$

Therefore,

$$\nabla p(x,t) = (\sigma W(t) - \mathbf{u})\cdot\nabla\mathbf{u}. \tag{2.5}$$

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(ii) Obviously, $\mathbf{u} = e^{\lambda\nu t}g(\xi)\mathbf{a}(x)$ satisfies the initial and boundary conditions. The rest of the proof follows from Lemma 2.1.

(iii) By Lemma 2.1, $\mathbf{u}(x,t;\xi) = e^{\lambda\nu(\xi)t}\mathbf{a}(x)$ satisfies

$$\mathbf{u}_t(x,t;\xi) = \lambda\nu(\xi)\mathbf{u}(x,t;\xi) = \nu(\xi)\Delta\mathbf{u}(x,t;\xi).$$

Further, as mentioned in the beginning of the proof, $\mathbf{u}(x,t;\xi)\cdot\nabla\mathbf{u}(x,t;\xi)$ is gradient.

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REFERENCES

[1] O. Walsh, *Eddy solutions of the Navier-Stokes equations*, The Navier-Stokes Equations II Theory and Numerical Methods, 1530 (1992), pp. 306-309.

[2] A. Chorin and O. H. Hald, *Stochastic Tools in Mathematics and Science* , Springer-Verlag, (2009).

[3] L. Berselli, *On the Large Eddy Simulation of the Taylor-Green vortex*, J. Math. Fluid Mech. 7(2005) S164-S191.

[4] W. Layton, *On Taylor/Eddy Solutions of Approximate Deconvolution Models of Turbulence*, Appl. Math. Lett., 24(1) (2011), pp. 23-26.

[5] G. I. Taylor, *On the decay of vortices in a visous fluid*, Phil. Mag., 46(6) (1923), pp. 671-674.

[6] G. I. Taylor and A. E. Green, *Mechanism of the Production of Small Eddies from Large Ones*, Proc. R. Soc. Lond. A, 158 (1937), pp. 499-521.

[7] H. N. Najm, *Uncertainty Quantification and Polynomial Chaos Techniques in Computational Fluid Dynamics*, Annu. Rev. Fluid Mech., 41 (2009), pp. 35-52.

[8] C. Webster, G. Zhang and M. Gunzburger, *An adaptive wavelet stochastic collocation method for irregular solutions of Stochastic Partial Differential Equations*, ORNL Tech. Rep., ORNL/TM-2012/186