Linear Algebra Preliminary Exam
May 10, 2026

1. Let $n \geq 2$ and $M_n(\mathbb{R})$ be the vector space of all $n \times n$ real matrices, and define $L : M_n(\mathbb{R}) \to M_n(\mathbb{R})$, $L(X) = X^{\mathsf{T}}$. Show that there does not exist a matrix $A \in M_n(\mathbb{R})$ such that $L(X) = AX$ for all $X \in M_n(\mathbb{R})$.

2. Let $n \geq 2$ and A be an $n \times n$ self-adjoint positive definite matrices. Show that
$$\det(I + A) \geq 1 + \det(A) + n\left[\det(A)\right]^{1/n}.$$

3. Let $T : V \to V$ be a linear operator on a finite-dimensional complex vector space V. Suppose that T is neither invertible nor nilpotent, show that there exists a decomposition $V = U \oplus W$, where both U and W are nontrivial T-invariant subspaces such that $T|_U$ is nilpotent, and $T|_W$ is invertible.

4. Suppose V is a linear space of polynomials with complex coefficients in real variables x and y defined by
$$V = \{ \sum_{0 \leq i,j \leq 2} a_{ij} x^i y^j | a_{ij} \in \mathbb{C} \}.$$
Suppose $T : V \to V$ is the partial derivative with respect to x, i.e. $Tp = \frac{\partial}{\partial x} p$ for any $p \in V$. Find the Jordan canonical form of T.

5. Let V be a real Euclidean space and $P : V \to V$ be a linear map satisfying $P^2 = P$. Suppose additionally that
$$\|Px\| \leq \|x\|$$
holds for every $x \in V$. Show that P is an orthogonal projection.

6. Let $A \in \mathbb{C}^{n \times n}$ satisfy $A^* A = I + A + A^*$, where A^* denotes the conjugate transpose of A, and I is the $n \times n$ identity matrix.
(a) Show that A is unitarily diagonalizable.
(b) Show that every eigenvalue λ of A satisfies $|\lambda - 1| = \sqrt{2}$.
(Hint: Consider the matrix $I - A$.)