Preliminary Exam in Analysis, May 2026

Problem 1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous. Assume that f is also differentiable in $\mathbb{R}^n \backslash \{0\}$, and that $\mathbb{R}^n \ni A = \lim_{x\to 0} Df(x)$ exists. Show that f is differentiable at 0 and $Df(0) = A$.

Problem 2. Let (M, d) be a compact metric space and $M \times M$ be given the metric $\rho((x, y), (u, v)) = d(x, u) + d(y, v)$ for $(x, y), (u, v) \in M \times M$. Let $f : M \times M \to \mathbb{R}$ be continuous on $M \times M$. For each $x \in M$ let

$$g(x) = \inf\{f(x, y) : y \in M\}.$$

Prove that g is continuous on M.

Problem 3. Let $g : (-1, 1) \to \mathbb{R}$ be continuous. For each $n \in \mathbb{N}$, let $f_n : (-1, 1)^2 \to \mathbb{R}$ be a continuously differentiable function on the square $(-1, 1)^2$. Assume that

$$f_n\left(\tfrac{n-1}{n}, \tfrac{1-n}{n}\right) = g(0)$$

and

$$|\nabla f_n(x, y)| \le \left|g\left(\tfrac{x+y}{2}\right)\right|^2 \qquad \text{for all } (x, y) \in (-1, 1)^2.$$

Prove that there is a continuous function $f : (-1, 1)^2 \to \mathbb{R}$ and a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ such that $f_{n_k} \to f$ uniformly on every compact subset of $(-1, 1)^2$.

Hint: *Prove first that the sequence $f_n(0, 0)$ is bounded.*

Problem 4. Given a positive integer n, prove that there exists a $\delta > 0$ such that for every $n \times n$ matrix A satisfying $\|A - I_n\| < \delta$, there exists an $n \times n$ matrix B such that $A^3 + B^3 = 2AB^T$, where B^T denotes the transpose of B, I_n is the $n \times n$ identity matrix, and for $A = (a_{ij})_{1\le i,j\le n}$,

$$\|A\| = \Big(\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2\Big)^{1/2}.$$

Problem 5. Prove that if $f : \mathbb{R} \to \mathbb{R}$ has the intermediate value property and the set $f^{-1}(\{q\})$ is closed for every rational q, then f is continuous.

Here, saying that f has the intermediate value property means that for every $a, b \in \mathbb{R}$ with $a < b$, and for every number y between $f(a)$ and $f(b)$, there exists $c \in [a, b]$ such that $f(c) = y$.

Problem 6. Let f be a polynomial of total degree at most three in $(x, y, z) \in \mathbb{R}^3$. Prove that

$$\int_{x^2+y^2+z^2\le 1} f(x, y, z)\, dx\, dy\, dz = \frac{4\pi f(0, 0, 0)}{3} + \frac{2\pi(\Delta f)(0, 0, 0)}{15}.$$

Here

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

is the Laplacian on $\mathbb{R}^3$.