Graduate Preliminary Examination in Linear Algebra

University of Pittsburgh

Instructions. Give complete proofs and justify each claim.

Problem 1. *Assume that A is a nilpotent matrix such that $\ker A^2 \subseteq \ker A$. Show that A is the zero matrix.*

Problem 2. *Let $T\colon V \to W$ be a linear map between finite-dimensional vector spaces. Define the dual map $T^*\colon W^* \to V^*$ as $T^* f = f \circ T$ for all linear functionals f on W.*

1. *Show that*
$$\ker T^* = (\operatorname{rng} T)^\perp, \qquad \operatorname{rng} T^* = (\ker T)^\perp,$$
where $U^\perp$ denotes the annihilator of U.

2. *Deduce that* $\operatorname{rank} T^* = \operatorname{rank} T$.

3. *Construct a natural isomorphism (i.e., one that does not depend on a choice of basis) between $V/\ker T$ and $\operatorname{rng} T$.*

Problem 3. *Assume that the numbers a_{ij} are unknown real quantities satisfying $-0.1 < a_{ij} < 0.1$ for $i, j \in \{1, \ldots, 5\}$. Show that the matrix below is both invertible and diagonalizable over $\mathbb{C}$.*

$$S = \begin{pmatrix} 1 + a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & 2 + a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & 3 + a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & 4 + a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & 5 + a_{55} \end{pmatrix}.$$

Problem 4. *Let $A, B \in M_n(\mathbb{C})$ be Hermitian.*

a) *Prove that if A and $AB + BA$ are positive definite, then B is positive definite.*

b) *Give an example to show that if A and $AB + BA$ are positive semidefinite, then B need not be positive semidefinite.*

Problem 5. *Let $A \in M_n(\mathbb{C})$ be diagonalizable and have a repeated eigenvalue. Prove that there exists a sequence of non-diagonalizable matrices converging to A, where convergence is defined with respect to any norm on $M_n(\mathbb{C})$.*

Problem 6. *Let $A \in M_n(\mathbb{C})$ be invertible and suppose that $\|A\| = \|A^{-1}\| = 1$, where $\|\cdot\|$ denotes the standard operator norm. Show that A is unitary.*