Linear Algebra Preliminary Exam May 2025

Problem 1 *Find the Jordan Canonical Form of*

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 4 \end{pmatrix}.$$

Note: *You do not have to perform the change of basis computations.*

Problem 2 *Let A, B be matrices, A invertible. Show that there exists $r_0 > 0$ such that $A - (1/r)\, B$ is invertible if $r > r_0$.*

Problem 3 *Let $n \geq 1$, and P_n be the linear space of all real coefficients polynomials of order at most n,*

$$P_n = \left\{ p\,(t) = \sum_{k=0}^{n} c_k t^k : c_k \in \mathbb{R}, 0 \leq k \leq n \right\}.$$

Show that there exist constants a_k, $0 \leq k \leq n$ such that $p\,(2024) = \sum_{k=0}^{n} a_k p\,(k)$ for every $p \in P_n$.

Problem 4 *Let A be a real $n \times n$ matrix. Show that the kernel of $A^T A$ is the same as the kernel of A.*

Problem 5 *Show that there exist no matrices $A \in \mathbb{R}^{3\times 2}$ and $B \in \mathbb{R}^{2\times 3}$ such that*

$$AB = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Problem 6 *Let $A \in \mathbb{C}^{n\times n}$ be a normal $n \times n$-matrix. Show that $\|A^n\| = \|A\|^n$ for the standard operator norm.*