Preliminary Exam in Analysis, August 2026

Problem 1. Let $f \in C^2(\mathbb{R}^2)$ be such that $f(0,0) = 0$. Prove that
$$\lim_{n\to\infty} \iint_D \frac{n^\pi f(x,y)}{1 + n^4(x^4 + y^4)}\, dA = 0,$$
where $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$ denotes the closed unit disc.
Hint: Use Taylor's theorem.

Problem 2. Suppose that $f, g : \mathbb{R}^2 \to \mathbb{R}$ are both C^1 in an open neighborhood U of $\mathbf{0} = (0,0)$ in $\mathbb{R}^2$ such that $f(\mathbf{0}) = g(\mathbf{0}) = 0$ and $\nabla f(\mathbf{0}) \cdot \nabla g(\mathbf{0}) \neq 0$. Prove that there exists a $\delta > 0$ such that for every $(\alpha, \beta) \in \mathbb{R}^2$ satisfying $|(\alpha,\, \beta)| < \delta$, there exists $(a,b) \in \mathbb{R}^2$ such that $f(a+b^2, a^2+b) = \alpha$ and $g(a^2 - b, a - b^2) = \beta$.

Problem 3. Let $\Lambda > 0$ and $r \in (1, \infty)$. Prove that there exists $\varepsilon_0 > 0$ with the following property: whenever $0 < \varepsilon < \varepsilon_0$ and $f : [0,\infty) \to [0,\infty)$ is continuous and satisfies
$$f(t) \le \Lambda f(t)^r + \varepsilon \qquad \text{for every } t \ge 0,$$
then
$$f(0) \le \varepsilon \quad \Longrightarrow \quad f(t) \le 2\varepsilon \qquad \text{for every } t \ge 0.$$

Problem 4. Let $f \in C^2(\mathbb{R}^n)$ satisfy $f(tx) = t^2 f(x)$ for every $x \in \mathbb{R}^n$ and every $t > 0$. Prove that f is a quadratic function. More precisely, prove that there is a symmetric matrix A such that
$$f(x) = \langle Ax, x\rangle \qquad \text{for every } x \in \mathbb{R}^n.$$

Problem 5. Let (X, d) be a compact metric space and let
$$\mathcal{F} = \{f \in C(X) : f(x) = \operatorname{dist}(x, K) \text{ for some nonempty compact set } K \subset X\}.$$
Prove that $\mathcal{F}$ is a compact subset of $C(X)$, where $C(X)$ is equipped with the supremum norm.

Remarks. (1) You can use the fact that $x \mapsto \operatorname{dist}(x, K)$ is 1-Lipschitz without proof. (2) It is not sufficient to prove that $\mathcal{F}$ is relatively compact. One must also prove that $\mathcal{F}$ is closed in $C(X)$, and hence compact.

Problem 6. Let $D \subset \mathbb{R}^d$, $d \ge 2$, be a compact convex set with smooth boundary ∂D, and assume that the origin does not belong to D. For every $x \in \partial D$, let $\alpha(x) \in [0, \pi]$ be the angle between the position vector x and the outer unit normal vector $\mathbf{n}(x)$. Let ω_d be the surface area of the unit sphere in $\mathbb{R}^d$. Compute
$$\frac{1}{\omega_d} \int_{\partial D} \frac{\cos(\alpha(x))}{|x|^{d-1}}\, dS(x).$$