Preliminary Exam in Analysis, May, 2025

1. Let $(a_k)_{k\in\mathbb{N}}$ be a strictly monotone increasing sequence with $a_k > 0$ for all k. Show that there exists a constant $C > 0$, depending only on a_1, such that the following inequality holds for all $k \in \mathbb{N}$:
$$\sum_{j=1}^{k}\left(1-\frac{a_j}{a_{j+1}}\right) \le \log a_k + C.$$

2. Let $f_k : [0,1] \to \mathbb{R}$ be a sequence of continuously differentiable functions and $f : [0,1] \to \mathbb{R}$, such that
   (1) for all $x \in [0,1]$, the sequence $(f_k(x))_{k\in\mathbb{N}}$ converges to $f(x)$, and
   (2) $\sup_{k\in\mathbb{N}} \int_{[0,1]} |f_k'(x)|^2 dx < \infty$.

   Show that $(f_k)_{k\in\mathbb{N}}$ (**not** just a subsequence) converges uniformly to f.

3. Let (X, d) be a nonempty, complete metric space and $f : X \to X$ (**not** necessarily continuous), let $N \in \mathbb{N}$ be a number, and let $\lambda \in (0,1)$ such that f^N satisfies
$$d\left(f^N(x), f^N(y)\right) \le \lambda\, d(x,y) \quad \forall x, y \in X.$$
   Here f^n denotes n-times application of f, e.g. $f^2(x) = f(f(x))$.

   Show that there exists $a \in X$ such that $\lim_{n\to\infty} f^n(x) = a$ for any $x \in X$. Also show that a is the unique fixed point of f.

4. Let $\{a_j\}_{j=1}^{\infty}$ be a sequence of real numbers, such that $\sum_j |a_j| = 1$ and $\sum_{j=1}^{\infty} a_j = \sum_{j=1}^{\infty} a_j^3 = 0$. Show that
$$\Big|\sum_{j=1}^{\infty}\sin(a_j)\Big| < \frac{1}{1000}.$$
   Hint: You could start by showing that $|a_j| \le \frac{1}{2}$ for each $j \in \mathbb{N}$.

5. Assume $n \ge 2$. Show that the following holds for any $f \in C^1(\mathbb{R}^n)$ which satisfies $f \equiv 0$ outside a compact set:
   (a) Fix any $x \in \mathbb{R}^n$, then
$$\sum_{i=1}^{n}\int_{\mathbb{R}^n\setminus B(x,\varepsilon)} \frac{x_i-y_i}{|x-y|^n}\,\partial_i f(y)\,dy = \varepsilon^{1-n}\int_{\partial B(x,\varepsilon)} f(y)dy.$$
   (b) Conclude that there exists a constant $C(n)$ such that
$$C(n)\,f(x) = \lim_{\varepsilon\to 0^+}\sum_{i=1}^{n}\int_{\mathbb{R}^n\setminus B(x,\varepsilon)} \frac{x_i-y_i}{|x-y|^n}\,\partial_i f(y)\,dy.$$

   Here, we denote vectors in $\mathbb{R}^n$ by $z = (z_1, z_2, ..., z_n)$, and the Euclidean norm $|z| = (\sum_{i=1}^{n}|z_i|^2)^{\frac{1}{2}}$.

6. Fix a **continuous** function $F : \mathbb{R} \to \mathbb{R}$ such that $F(0) \ne 0$. Show that for some $\varepsilon > 0$ the problem
$$\begin{cases} \dfrac{dy}{dt} = F(y) & t \in (-\varepsilon, \varepsilon) \\ y(0) = 0 \end{cases}$$
   has a unique solution $y \in C^1((-\varepsilon,\varepsilon),\mathbb{R})$.

   Hint: Formally consider a solution to the differential equation $\frac{dt}{dy} = \frac{1}{F(y)}$.