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ENERGY STABILITY OF A FIRST ORDER PARTITIONED METHOD FOR SYSTEMS WITH GENERAL COUPLING

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Abstract. We give an energy stability analysis of a first order, 2 step partitioned time discretization of systems of evolution equations. The method requires only uncoupled solutions of sub-systems at every time step without iteration, is long time stable and applies to general system couplings. We give a proof of long time energy stability under a time step restriction relating the time step to the size of the coupling terms.

Key words. partitioned methods, energy stability.

1. Introduction

The most natural approach to numerical simulation of multi-domain, multi-physics systems is to build a partitioned method for the system from components optimized for the individual sub-domain and sub-physics problems. The two most common approaches to partitioning are implicit-explicit methods where the system's' coupling terms are discretized by explicit methods and sub-domain / sub-physics terms by implicit methods, and splitting methods where the coupling terms are themselves separated in each equation according to the subproblems. Application of either to complex problems requires analytic foundations as a guide for practical computation. Herein, we consider the first, implicit-explicit, approach for *general couplings* (the main point herein) but restricted to first order, two step methods. Thus, for a system

$$\begin{aligned}\frac{d}{dt}u_1 + A_1u_1 + B_{11}u_1 + B_{12}u_2 &= f_1,\\ \frac{d}{dt}u_2 + A_2u_2 + B_{21}u_1 + B_{22}u_2 &= f_2,\end{aligned} \tag{1.1}$$

we analyze long-time, energy stability of a method (1.6) below which requires at each step that the two uncoupled linear systems be solved

$$[I + 2\triangle tA_i]u_i^{n+1} = RHS,\ \ i = 1, 2. \tag{1.2}$$

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Thus, we consider methods implicit in A_1u_1 and A_2u_2 but explicit in the coupling terms $B_{11}u_1 + B_{12}u_2$ and $B_{21}u_1 + B_{22}u_2$. In the method, the diagonal terms $B_{11}u_1$ and $B_{22}u_2$ could be incorporated into the part treated implicitly (the A_i's). However, the part treated implicitly in (1.3), (1.1) is often determined by existing codes and the coupling terms are those that remain.

Letting $u = (u_1, u_2)^T : [0, \infty) \to \mathbb{R}^d$ and A, B represent the $d \times d$ block matrices in (1.1), we develop the stability analysis for

$$\frac{du}{dt} + Au + Bu = f, u(0) = u_0. \tag{1.3}$$

Let $\langle \cdot, \cdot \rangle, || \cdot ||$ denote the Euclidean inner product and norm. Suppose

$$A > 0 \text{ i.e., } \langle Au, u\rangle > 0 \text{ for all } u \in \mathbb{R}^d.$$

Partitioned methods, herein, are useful tools and not best for every circumstance. The equally useful alternative is a monolithic method where the fully coupled system is assembled and solved by some iterative method wherein uncoupling can occur in the preconditioning step, e.g., [9] for one example. Conservative couplings ($B^* = -B$, where B^* satisfies $\langle Bx, y\rangle = \langle x, B^*y\rangle \ \forall x, y$) occur when what is lost to one domain or variable is transferred to the other. One important example is the evolutionary Stokes-Darcy model under the Beaver-Joseph-Saffman-Jones (BJSJ) interface condition, [18], [14], [15], [10], [13], [12]. Dissipative couplings ($B = B^* > 0$) occur when there is energy lost through the interaction of the two systems. One important example is in atmosphere-ocean couplings under the rigid lid condition under which there are frictional lossess in transmitting wind energy at the ocean surface to the ocean (and vice versa), e.g., [3], [5], [6], [7]. Another important example of dissipative couplings is compressible flow in a porous medium. The double porosity model of slightly compressible flow in a porous medium [17] is: find $u_1(x, t), u_2(x, t)$

$$\begin{aligned}
(c_1u_1)_t - \frac{k_1c_0}{\mu}\triangle u_1 + \alpha^{-1}(u_1 - u_2) &= f,\\
(c_2u_2)_t - \frac{k_2c_0}{\mu}\triangle u_2 + \alpha^{-1}(u_2 - u_1) &= g.
\end{aligned}$$

The coupling term satisfies

$$\begin{aligned}
(B\mathbf{u}, \mathbf{u}) &= \int_{\text{flow domain}} \alpha^{-1}(u_1 - u_2)u_1 + \alpha^{-1}(u_2 - u_1)u_2dx\\
&= \alpha^{-1} \int_{\text{flow domain}} (u_1 - u_2)^2dx \geq 0
\end{aligned}$$

and is thus dissipative.

One important case where dissipative, conservative and resonant couplings dominated by system dissipation are present is the (above discussed) Stokes-Darcy problem under the orginal (BJ) Beavers-Joesph condition, studied in [4], [16]. Compared to the BJSJ condition, extra terms occur which are resonant and must be sufficiently small in the theory developed in these papers. Building on this previous work, we give combination of these treatments stable for general couplings. The method we study is related to work in [1], [14] and of implicit-explicit type, e.g., [2], [8]. The analytical treatments of component discretizations of the coupling terms in (1.3), (1.1) are known (by a different analytical path for each type of coupling). However, the analysis of energy stability of their combination presents the technical difficulty that one discrete evolution equation with all type present requires one analytical path.

We decompose $B = C+P-N$ (skew symmetric, symmetric positive and symmetric negative parts) and use explicit time discretizations suggested by linear stability theory for each part. Let

$$\left\{ \begin{array}{c} B = C + P - N \text{ where,} \\ C^* = -C, \quad P^* = P \geq 0, \quad \text{and} \quad N^* = N \geq 0. \end{array} \right. \tag{1.4}$$

The coupling term Cu is conservative, the term Pu is dissipative, the term $-Nu$ is resonant. For long time stability of (1.1), the resonant coupling must be dominated by the sub-system dissipation. Thus, we assume

$$A > 0 \quad \text{and} \quad A - N \geq a_0 I > 0. \tag{1.5}$$

These assumptions imply the basic stability properties sought to be preserved under discretization:

$$\sup_{[0,\infty)} ||u(t)|| \leq ||u_0|| + C \sup_{[0,\infty)} ||f(t)||$$
$$\text{and if} \quad f = 0,$$
$$||u(t)||^2 \to 0 \text{ as } t \to \infty.$$

1.1. The Method. We thus consider a method that is a combination of Backward Euler - Leapfrog - Forward Euler for the components of (1.3): Given u^0 and (calculated by some other method to appropriate accuracy [19]) $u^1 \in X$ find $u^n \in X$ for $n \geq 2$ satisfying

$$\frac{u^{n+1} - u^{n-1}}{2\triangle t} + Au^{n+1} + Cu^n + (P - N)\, u^{n-1} = f(t^{n+1}). \tag{1.6}$$

This is a 2 step method due to the use of leap frog for the skew symmetric, coupling term. Approximations are needed at the first time step to appropriate accuracy, [19]. For (1.1) above, discretized in time by (1.6), at each step, the uncoupled linear systems (1.2) must be solved.

2. Energy Stability

We prove long time, energy stability of the uncoupling method (1.6) under time step conditions (2.1) or (2.2) below. For a self-adjoint, positive semi-definite matrix

Q, we denote the induced semi-norm and semi-inner product by

$$\langle u,v\rangle_Q := \langle Qu,v\rangle, ||u||_Q := \sqrt{\langle u,v\rangle_Q}.$$

Stability requires a time step condition which will be independent of $||N||$ due to (1.5). We suppose that, for some $\alpha>0$, there holds *either*

$$\begin{aligned} \triangle t||C|| &\leq 1-\alpha<1,\\ 4\triangle t||P|| &\leq 1-\alpha<1,\\ 2\triangle t||C|| &\leq (1-\alpha)a_0<a_0. \end{aligned} \tag{2.1}$$

or

$$\triangle t(||P||+||C||)\leq 1-\alpha<1. \tag{2.2}$$

Theorem 1 (Energy Stability)**.** *Consider the discretization (1.6) of (1.3) under the structure conditions (1.4), (1.5). Suppose that $\triangle t$ satisfies either (2.1) or (2.2).*

If (2.1) holds, then

$$\begin{aligned} &\alpha\left(||u^N||^2+||u^{N-1}||^2\right)+2\triangle t||u^N||_A^2+2\triangle t||u^{N-1}||_A^2+\triangle t||u^N||_{A-N}^2\\ &+\sum_{n=1}^{N-1}\frac{\alpha}{2}||u^{n+1}-u^{n-1}||^2+2\triangle t\sum_{n=1}^{N-1}||u^{N-1}||_{A-N}^2\\ &+\triangle t\sum_{n=1}^{N-1}\left[\frac{a_0}{2}||u^{n+1}||^2+\alpha a_0||u^n||^2+2||u^{n+1}||_P^2+2||u^{n-1}||_P^2+2||u^{n+1}-u^{n-1}||_N^2\right]\\ &\leq(||u^1||^2+||u^0||^2+2\triangle t\left\langle Cu^0,u^1\right\rangle+2\triangle t||u^0||_A^2+2\triangle t||u^1||_A^2+\triangle t||u^1||_{A-N}^2)+\\ &+\triangle t\sum_{n=1}^{N-1}\frac{8}{a_0}||f(t^{n+1})||^2. \end{aligned} \tag{2.3}$$

If (2.2) holds, then

$$\begin{aligned} &\alpha\left(||u^N||^2+||u^{N-1}||^2\right)+\triangle t||u^N||_{A+N}^2+\triangle t||u^{N-1}||_{A+N}^2\\ &+\triangle t\sum_{n=1}^{N-1}||u^{n+1}+u^{n-1}||_{P+Ia_0/2}^2\\ &\leq(||u^1||_{I-\triangle tP}^2+||u^0||_{I-\triangle tP}^2)+\triangle t|\left\langle Cu^0,u^1\right\rangle|+\triangle t||u^0||_{A+N}^2+\triangle t||u^1||_{A+N}^2\\ &+\triangle t\sum_{n=1}^{N-1}\frac{2}{a_0}||f(t^{n+1})||^2. \end{aligned} \tag{2.4}$$

If $f(t)\equiv 0$ for t large enough, or more generally if $\sum_{n=1}^{\infty}||f(t^{n+1})||^2<\infty$, we have

$$\begin{aligned} u^n &\to 0 \quad \textit{as} \quad t_n\to\infty \ \textit{under (2.1)},\\ u^{n+1}+u^{n-1} &\to 0 \quad \textit{as} \quad t_n\to\infty \ \textit{under (2.2)}. \end{aligned}$$

Proof. **The case of $\triangle t$ condition (2.1).** The key will be the treatment of the coupling terms. Before addressing them, we begin with a few standard steps. Multiply through by $4\triangle t$, take the inner with u^{n+1}, use the polarization identity for the term $\langle u^{n+1}, u^{n-1}\rangle$ and add and subtract $||u^n||^2$. This gives

$$\begin{gathered}(||u^{n+1}||^2+||u^n||^2)-(||u^n||^2+||u^{n-1}||^2)+||u^{n+1}-u^{n-1}||^2\\ +4\triangle t||u^{n+1}||_A^2+\\ +4\triangle t\left\langle Cu^n,u^{n+1}\right\rangle+4\triangle t\left\langle (P-N)u^{n-1},u^{n+1}\right\rangle=4\triangle t\left\langle f(t^{n+1}),u^{n+1}\right\rangle.\end{gathered}$$

Since $P\geq 0$ we apply the polarization identity to the term $\langle Pu^{n-1},u^{n+1}\rangle$

$$\left\langle Pu^{n-1},u^{n+1}\right\rangle=\frac{1}{2}||u^{n+1}||_P^2+\frac{1}{2}||u^{n-1}||_P^2-\frac{1}{2}||u^{n+1}-u^{n-1}||_P^2.$$

Rewrite the skew coupling term as

$$4\triangle t\left\langle Cu^n,u^{n+1}\right\rangle=2\triangle t\left\langle Cu^n,u^{n+1}\right\rangle-2\triangle t\left\langle Cu^{n-1},u^n\right\rangle+2\triangle t\left\langle Cu^n,u^{n+1}-u^{n-1}\right\rangle.$$

The last term on the RHS satisfies

$$|2\triangle t\left\langle Cu^n,u^{n+1}-u^{n-1}\right\rangle|\leq\frac{1}{2}||u^{n+1}-u^{n-1}||^2+2\triangle t^2||Cu^n||^2. \tag{2.5}$$

Combining and rearranging gives

$$\begin{gathered}(||u^{n+1}||^2+||u^n||^2+2\triangle t\left\langle Cu^n,u^{n+1}\right\rangle)-(||u^n||^2+||u^{n-1}||^2+2\triangle t\left\langle Cu^{n-1},u^n\right\rangle)+\\ +\left[\frac{1}{2}||u^{n+1}-u^{n-1}||^2-2\triangle t||u^{n+1}-u^{n-1}||_P^2\right]\\ +4\triangle t||u^{n+1}||_A^2+2\triangle t||u^{n+1}||_P^2+2\triangle t||u^{n-1}||_P^2-2\triangle t^2||Cu^n||^2\\ -4\triangle t\left\langle Nu^{n-1},u^{n+1}\right\rangle\leq 4\triangle t\left\langle f(t^{n+1}),u^{n+1}\right\rangle.\end{gathered} \tag{2.6}$$

In the balance between the global dissipation and the resonant coupling, the polarization identity on the N semi-inner product:

$$\begin{gathered}||u^{n+1}||_A^2-\left\langle Nu^{n-1},u^{n+1}\right\rangle=\\ =||u^{n+1}||_A^2-\left[\frac{1}{2}||u^{n-1}||_N^2+\frac{1}{2}||u^{n+1}||_N^2-\frac{1}{2}||u^{n+1}-u^{n-1}||_N^2\right]\\ =\frac{1}{2}\left\langle (A-N)u^{n+1},u^{n+1}\right\rangle+\frac{1}{2}||u^{n+1}-u^{n-1}||_N^2+\frac{1}{2}\left[||u^{n+1}||_A^2-||u^{n-1}||_N^2\right]\\ =\frac{1}{2}\left\langle (A-N)u^{n+1},u^{n+1}\right\rangle+\frac{1}{2}||u^{n+1}-u^{n-1}||_N^2\\ +\frac{1}{2}\left[||u^{n+1}||_A^2-\{||u^{n-1}||_A^2-\left\langle (A-N)u^{n-1},u^{n-1}\right\rangle\}\right]\\ =\frac{1}{2}\left\langle (A-N)u^{n+1},u^{n+1}\right\rangle+\frac{1}{2}\left\langle (A-N)u^{n-1},u^{n-1}\right\rangle\\ +\frac{1}{2}||u^{n+1}-u^{n-1}||_N^2+\frac{1}{2}\left[||u^{n+1}||_A^2-||u^{n-1}||_A^2\right].\end{gathered}$$

Insert this for the corresponding terms in (2.6) and add and subtract $2\triangle t||u^n||_A^2$. We obtain (recalling that $A - N > 0$)

$$\begin{gathered}
[||u^{n+1}||^2 + ||u^n||^2 + 2\triangle t \left\langle Cu^n, u^{n+1}\right\rangle + 2\triangle t||u^{n+1}||_A^2 + 2\triangle t||u^n||_A^2] \\
-[||u^n||^2 + ||u^{n-1}||^2 + 2\triangle t \left\langle Cu^{n-1}, u^n\right\rangle + 2\triangle t||u^{n-1}||_A^2 + 2\triangle t||u^n||_A^2] \\
+ \left[\frac{1}{2}||u^{n+1} - u^{n-1}||^2 - 2\triangle t||u^{n+1} - u^{n-1}||_P^2 + 2\triangle t||u^{n+1} - u^{n-1}||_N^2\right] \\
+2\triangle t \left(||u^{n+1}||_{A-N}^2 + ||u^{n-1}||_{A-N}^2 - \triangle t||Cu^n||^2\right) \\
+2\triangle t||u^{n+1}||_P^2 + 2\triangle t||u^{n-1}||_P^2 \leq 4\triangle t \left\langle f(t^{n+1}), u^{n+1}\right\rangle.
\end{gathered}$$

The following term requires more analysis:

$$\begin{gathered}
2\triangle t \left(||u^{n+1}||_{A-N}^2 + ||u^{n-1}||_{A-N}^2 - \triangle t||Cu^n||^2\right) = \\
= \triangle t[||u^{n+1}||_{A-N}^2 + \langle (A-N)u^n, u^n\rangle - 2\triangle t||Cu^n||^2] + \\
+\triangle t[||u^{n+1}||_{A-N}^2 - ||u^n||_{A-N}^2] + 2\triangle t\|u^{n-1}\|_{A-N}^2.
\end{gathered}$$

Since $2\triangle t||C|| \leq (1-\alpha)a_0$ we have

$$\langle (A-N)u^n, u^n\rangle - 2\triangle t||Cu^n||^2 \geq \alpha a_0||u^n||^2.$$

These identities give $E^{n+1/2}$-$E^{n-1/2}$ + {positive} $\leq 4\triangle t \left\langle f(t^{n+1}), u^{n+1}\right\rangle$:

$$\begin{gathered}
\left[||u^{n+1}||^2 + ||u^n||^2 + 2\triangle t\left(\left\langle Cu^n, u^{n+1}\right\rangle + ||u^{n+1}||_A^2 + ||u^n||_A^2 + \frac{1}{2}||u^{n+1}||_{A-N}^2\right)\right] \\
- \left[||u^n||^2 + ||u^{n-1}||^2 + 2\triangle t\left(\left\langle Cu^{n-1}, u^n\right\rangle + ||u^{n-1}||_A^2 + ||u^n||_A^2 + \frac{1}{2}||u^n||_{A-N}^2\right)\right] \\
+\left\{\frac{1}{2}||u^{n+1} - u^{n-1}||^2 - 2\triangle t||u^{n+1} - u^{n-1}||_P^2 + 2\triangle t||u^{n+1} - u^{n-1}||_N^2\right. \\
+\triangle t||u^{n+1}||_{A-N}^2 + 2\triangle t||u^{n-1}||_{A-N}^2 + \triangle t\alpha a_0\|u^n\|^2 \\
\left. +2\triangle t||u^{n+1}||_P^2 + 2\triangle t||u^{n-1}||_P^2\right\} \leq 4\triangle t\left\langle f(t^{n+1}), u^{n+1}\right\rangle.
\end{gathered} \tag{2.7}$$

The energy $E^{n+1/2}$ is positive provided $\triangle t||C|| < 1 - \alpha$:

$$\begin{gathered}
||u^{n+1}||^2 + ||u^n||^2 + 2\triangle t\left(\left\langle Cu^n, u^{n+1}\right\rangle + ||u^{n+1}||_A^2 + ||u^n||_A^2 + \frac{1}{2}||u^{n+1}||_{A-N}^2\right) \\
\geq ||u^{n+1}||^2 + ||u^n||^2 - 2\triangle t||C||||u^n||||u^{n+1}|| + \\
+2\triangle t\left(||u^{n+1}||_A^2 + ||u^n||_A^2 + \frac{1}{2}||u^{n+1}||_{A-N}^2\right) \\
\geq \alpha\left(||u^{n+1}||^2 + ||u^n||^2\right) + 2\triangle t\left(||u^{n+1}||_A^2 + ||u^n||_A^2\right) + \triangle t||u^{n+1}||_{A-N}^2.
\end{gathered}$$

The third line in (2.7) is positive under $4\triangle t||P|| \leq 1-\alpha$:

$$\left[\frac{1}{2}||u^{n+1}-u^{n-1}||^2 - 2\triangle t||u^{n+1}-u^{n-1}||_P^2 + 2\triangle t||u^{n+1}-u^{n-1}||_N^2\right] \geq$$
$$= \frac{1}{2}\left\langle (I-4\triangle tP)\left(u^{n+1}-u^{n-1}\right), u^{n+1}-u^{n-1}\right\rangle + 2\triangle t||u^{n+1}-u^{n-1}||_N^2 \geq$$
$$\geq \frac{\alpha}{2}||u^{n+1}-u^{n-1}||^2 + 2\triangle t||u^{n+1}-u^{n-1}||_N^2 \geq 0.$$

Inserting these lower estimates we have

$$\left[||u^{n+1}||^2 + ||u^n||^2 + 2\triangle t\left(\left\langle Cu^n, u^{n+1}\right\rangle + ||u^{n+1}||_A^2 + ||u^n||_A^2 + \frac{1}{2}||u^{n+1}||_{A-N}^2\right)\right]$$
$$-\left[||u^n||^2 + ||u^{n-1}||^2 + 2\triangle t\left(\left\langle Cu^{n-1}, u^n\right\rangle + ||u^{n-1}||_A^2 + ||u^n||_A^2 + \frac{1}{2}||u^n||_{A-N}^2\right)\right]$$
$$+\frac{\alpha}{2}||u^{n+1}-u^{n-1}||^2 + 2\triangle t||u^{n+1}-u^{n-1}||_N^2 + \triangle t(2||u^{n-1}||_{A-N}^2$$
$$+a_0||u^{n+1}||^2 + \alpha a_0||u^n||^2 + 2||u^{n+1}||_P^2 + 2||u^{n-1}||_P^2) \leq 4\triangle t\left\langle f(t^{n+1}), u^{n+1}\right\rangle.$$

Consider now the RHS

$$4\triangle t\left\langle f(t^{n+1}), u^{n+1}\right\rangle \leq \triangle t\frac{a_0}{2}||u^{n+1}||^2 + \triangle t\frac{8}{a_0}||f(t^{n+1})||^2.$$

Thus,

$$\left[||u^{n+1}||^2 + ||u^n||^2 + 2\triangle t\left(\left\langle Cu^n, u^{n+1}\right\rangle + ||u^{n+1}||_A^2 + ||u^n||_A^2 + \frac{1}{2}||u^{n+1}||_{A-N}^2\right)\right]$$
$$-\left[||u^n||^2 + ||u^{n-1}||^2 + 2\triangle t\left(\left\langle Cu^{n-1}, u^n\right\rangle + ||u^{n-1}||_A^2 + ||u^n||_A^2 + \frac{1}{2}||u^n||_{A-N}^2\right)\right]$$
$$+\frac{\alpha}{2}||u^{n+1}-u^{n-1}||^2 + 2\triangle t||u^{n+1}-u^{n-1}||_N^2 + 2\triangle t(|u^{n-1}||_{A-N}^2$$
$$+\frac{a_0}{4}||u^{n+1}||^2 + \frac{\alpha a_0}{2}||u^n||^2 + ||u^{n+1}||_P^2 + ||u^{n-1}||_P^2) \leq \triangle t\frac{8}{a_0}||f(t^{n+1})||^2.$$

Summing $n = 1, \cdots, N-1$ and the lower bound for E gives

$$\alpha\left(||u^N||^2 + ||u^{N-1}||^2\right) + 2\triangle t||u^N||_A^2 + 2\triangle t||u^{N-1}||_A^2 + \triangle t||u^N||_{A-N}^2$$
$$+\sum_{n=1}^{N-1}\frac{\alpha}{2}||u^{n+1}-u^{n-1}||^2 + 2\triangle t\sum_{n=1}^{N-1}||u^{n+1}-u^{n-1}||_N^2$$
$$+\triangle t\sum_{n=1}^{N-1}\left[2||u^{N-1}||_{A-N}^2 + \frac{a_0}{2}||u^{n+1}||^2 + \alpha a_0||u^n||^2 + 2||u^{n+1}||_P^2 + 2||u^{n-1}||_P^2\right]$$
$$\leq (||u^1||^2 + ||u^0||^2 + 2\triangle t|\left\langle Cu^0, u^1\right\rangle| + 2\triangle t||u^0||_A^2 + 2\triangle t||u^1||_A^2 + \triangle t||u^1||_{A-N}^2)+$$
$$+\triangle t\sum_{n=1}^{N-1}\frac{8}{a_0}||f(t^{n+1})||^2.$$

The case of $\triangle t$ condition (2.2). Take the inner product of (1.6) with $u^{n+1}+u^{n-1}$, rewrite the skew coupling terms as

$$\left\langle Cu^n, u^{n+1}+u^{n-1}\right\rangle = \left\langle Cu^n, u^{n+1}\right\rangle - \left\langle Cu^{n-1}, u^n\right\rangle.$$

This gives

$$\begin{aligned}\frac{||u^{n+1}||^2-||u^{n-1}||^2}{2\triangle t} &+ \left[\left\langle Cu^n, u^{n+1}\right\rangle - \left\langle Cu^{n-1}, u^n\right\rangle\right]\\ +\left[\left\langle Au^{n+1}, u^{n+1}+u^{n-1}\right\rangle \right.&\left.+ \left\langle (P-N)u^{n-1}, u^{n+1}+u^{n-1}\right\rangle\right]\\ &= \left\langle f(t^{n+1}), u^{n+1}+u^{n-1}\right\rangle.\end{aligned}$$

If we add, subtract $\left\langle Au^{n-1}, u^{n-1}+u^{n+1}\right\rangle$ to the last bracketed terms and use the fact that $\left\langle A(u^{n+1}-u^{n-1}), u^{n-1}+u^{n+1}\right\rangle = \|u^{n+1}\|_A^2 - \|u^{n-1}\|_A^2$, we obtain

$$\begin{aligned}&\left\langle Au^{n+1}, u^{n+1}+u^{n-1}\right\rangle + \left\langle (P-N)u^{n-1}, u^{n+1}+u^{n-1}\right\rangle\\ &\quad= \|u^{n+1}\|_A^2 - \|u^{n-1}\|_A^2 + \left\langle Au^{n-1}, u^{n-1}+u^{n+1}\right\rangle\\ &\qquad + \left\langle (P-N)u^{n-1}, u^{n+1}+u^{n-1}\right\rangle.\end{aligned}$$

Applying the polarization identity to $\left\langle Pu^{n-1}, u^{n+1}+u^{n-1}\right\rangle$ gives

$$\langle Pu^{n-1}, u^{n-1}+u^{n+1}\rangle = \frac{1}{2}||u^{n-1}||_P^2 + \frac{1}{2}||u^{n+1}+u^{n-1}||_P^2 - \frac{1}{2}||u^{n+1}||_P^2.$$

Thus, we have that

$$\begin{aligned}&\left\langle Au^{n+1}, u^{n+1}+u^{n-1}\right\rangle + \left\langle (P-N)u^{n-1}, u^{n+1}+u^{n-1}\right\rangle\\ &= \|u^{n+1}\|_A^2 - \|u^{n-1}\|_A^2 + \left\langle (A-N)u^{n-1}, u^{n-1}+u^{n+1}\right\rangle\\ &\quad + \frac{1}{2}||u^{n-1}||_P^2 + \frac{1}{2}||u^{n+1}+u^{n-1}||_P^2 - \frac{1}{2}||u^{n+1}||_P^2.\end{aligned}$$

Similarly, the second term satisfies

$$\left\langle (A-N)u^{n-1}, u^{n-1}+u^{n+1}\right\rangle = \frac{1}{2}\left(||u^{n-1}||_{A-N}^2 - ||u^{n+1}||_{A-N}^2\right) + \frac{1}{2}||u^{n+1}+u^{n-1}||_{A-N}^2.$$

Combining and rearranging gives

$$\begin{aligned}&\left\langle Au^{n+1}, u^{n+1}+u^{n-1}\right\rangle + \left\langle (P-N)u^{n-1}, u^{n+1}+u^{n-1}\right\rangle\\ &\quad= \frac{1}{2}\left(||u^{n+1}+u^{n-1}||_{A-N}^2 + ||u^{n+1}+u^{n-1}||_P^2\right)\\ &+\frac{1}{2}\left(||u^{n+1}||_{A+N}^2 - ||u^{n-1}||_{A+N}^2\right) + \frac{1}{2}\left(||u^{n-1}||_P^2 - ||u^{n+1}||_P^2\right).\end{aligned}$$

Using $A-N \geq a_0 I > 0$ and incorporating above calculations, we have

$$\begin{aligned}E^{n+1/2} - E^{n-1/2} &+ \triangle t||u^{n+1}+u^{n-1}||_P^2 + a_0\triangle t||u^{n+1}+u^{n-1}||^2\\ &\leq 2\triangle t\left\langle f(t^{n+1}), u^{n+1}+u^{n-1}\right\rangle.\end{aligned}$$

where the energy

$$\begin{aligned}E^{n+1/2} &= ||u^{n+1}||_{I-\triangle tP}^2 + ||u^n||_{I-\triangle tP}^2 + 2\triangle t\left\langle Cu^n, u^{n+1}\right\rangle\\ &+ \triangle t\left(||u^{n+1}||_{A+N}^2 + ||u^n||_{A+N}^2\right)\end{aligned}$$

is positive provided $\triangle t(||C|| + ||P||) \leq 1 - \alpha < 1$. Indeed,

$$\begin{gathered}||u^{n+1}||^2_{I-\triangle tP} + ||u^n||^2_{I-\triangle tP} + 2\triangle t\left\langle Cu^n, u^{n+1}\right\rangle + \triangle t\left(||u^{n+1}||^2_{A+N} + ||u^n||^2_{A+N}\right) \geq \\ ||u^{n+1}||^2_{I-\triangle tP} + ||u^n||^2_{I-\triangle tP} - 2\triangle t\|C\|\|u^n\|\|u^{n+1}\| + \triangle t\left(||u^{n+1}||^2_{A+N} + ||u^n||^2_{A+N}\right) \geq \\ \alpha\left(||u^{n+1}||^2 + ||u^n||^2\right) + \triangle t\left(||u^{n+1}||^2_{A+N} + ||u^n||^2_{A+N}\right) \geq 0.\end{gathered}$$

Consider now the RHS

$$2\triangle t\left\langle f(t^{n+1}), u^{n+1} + u^{n-1}\right\rangle \leq \triangle t\left(a_0\frac{||u^{n+1} + u^{n-1}||^2}{2} + \frac{2}{a_0}||f(t^{n+1})||^2\right).$$

Thus

$$\begin{gathered}[||u^{n+1}||^2_{I-\triangle tP} + ||u^n||^2_{I-\triangle tP} + \triangle t\left(2\left\langle Cu^n, u^{n+1}\right\rangle + ||u^{n+1}||^2_{A+N} + ||u^n||^2_{A+N}\right)] \\ -[||u^n||^2_{I-\triangle tP} + ||u^{n-1}||^2_{I-\triangle tP} + \triangle t\left(2\left\langle Cu^{n-1}, u^n\right\rangle + ||u^n||^2_{A+N} + ||u^{n-1}||^2_{A+N}\right)] \\ +\triangle t||u^{n+1} + u^{n-1}||^2_{P+Ia_0/2} \leq \triangle t\frac{2}{a_0}||f(t^{n+1})||^2.\end{gathered}$$

Summing from $n = 1, \cdot\cdot\cdot, N-1$ and using the lower bound for the system energy gives

$$\begin{gathered}\alpha\left(||u^N||^2 + ||u^{N-1}||^2\right) + \triangle t||u^N||^2_{A+N} + \triangle t||u^{N-1}||^2_{A+N} \\ +\triangle t\sum_{n=1}^{N-1}||u^{n+1} + u^{n-1}||^2_{P+Ia_0/2} \\ \leq (||u^1||^2_{I-\triangle tP} + ||u^0||^2_{I-\triangle tP}) + 2\triangle t|\left\langle Cu^0, u^1\right\rangle| + \triangle t||u^0||^2_{A+N} + \triangle t||u^1||^2_{A+N} \\ +\triangle t\sum_{n=1}^{N-1}\frac{2}{a_0}||f(t^{n+1})||^2.\end{gathered}$$

Asymptotic stability. For the claim that $u^n \to 0$ as $n \to \infty$, set $f = 0$ and simply drop some non-negative terms from the LHS. We then have, in cases 1 and 2 respectively,

$$\begin{aligned}\left(||u^N||^2 + ||u^{N-1}||^2\right) + \triangle t\sum_{n=1}^{N-1}\left[||u^{n+1}||^2 + ||u^n||^2\right] &< \infty, \\ \alpha\left(||u^N||^2 + ||u^{N-1}||^2\right) + \frac{a_0\triangle t}{2}\sum_{n=1}^{N-1}\left[||u^{n+1} + u^{n-1}||^2\right] &< \infty.\end{aligned}$$

Since each RHS is independent of N, the sums $\sum_{n=1}^{\infty}[\cdot]$ converge and thus the n^{th} terms $\to 0$. □

3. Numerical experiment

We consider a simple example. Let $T = 1$, $f = (-1, 0)^T$ and

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}, N = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 0 & -50 \\ 50 & 0 \end{pmatrix},$$
$$P = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}, u^0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, u^1 = \begin{pmatrix} 1.1 \\ 0.9 \end{pmatrix}.$$

The true solution is given by

$$u = \begin{pmatrix} \frac{8382495e^{-\frac{7x}{2}}\cos\frac{3\sqrt{1111}x}{2} - 9999\cos\left[\frac{3\sqrt{1111}x}{2}\right]^2 + 243685\sqrt{1111}e^{-\frac{7x}{2}}\sin\frac{3\sqrt{1111}x}{2} - 9999\sin\left[\frac{3\sqrt{1111}x}{2}\right]^2}{8372496} \\ \frac{4102923e^{-\frac{7x}{2}}\cos\frac{3\sqrt{1111}x}{2} + 83325\cos\left[\frac{3\sqrt{1111}x}{2}\right]^2 - 124519\sqrt{1111}e^{-\frac{7x}{2}}\sin\frac{3\sqrt{1111}x}{2} + 83325\sin\left[\frac{3\sqrt{1111}x}{2}\right]^2}{4186248} \end{pmatrix},$$

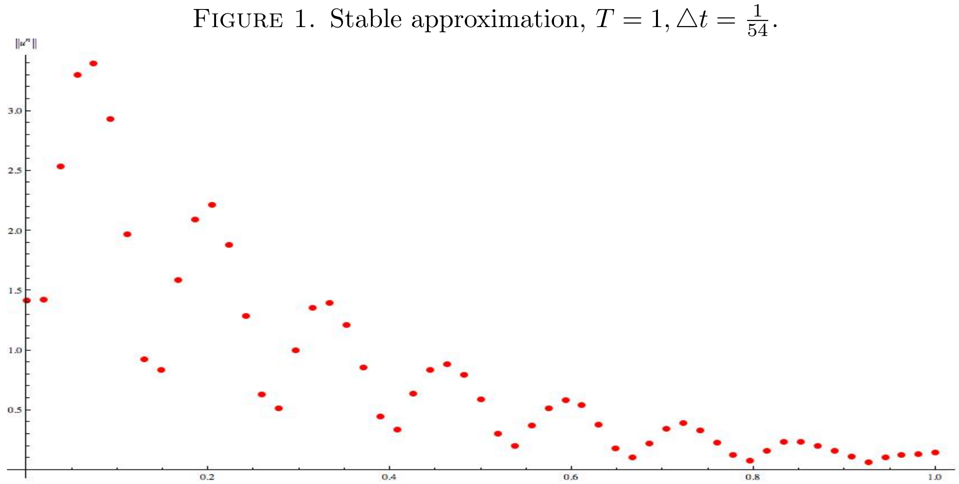
which was computed using *Mathematica* command *DSolve*.

With the above choices of matrices, we have $\|C\|_2 = 50$, $\|P\|_2 = 3$. Since $a_0 = 1$, the two time step restrictions (2.1),(2.2) are, respectively,

$$\triangle t < 1/\min\{2||C||, 4||P||\} = \frac{1}{100},$$
$$\triangle t < 1/(||C|| + ||P||) = \frac{1}{53}$$

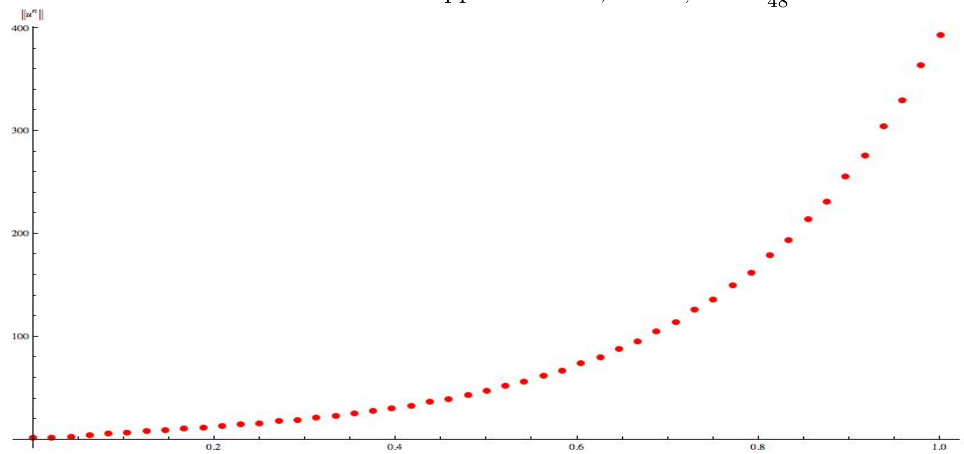
We performed two tests. As expected, for $\triangle t = \frac{1}{54}$, the solution is stable (Fig. 1) and for $\triangle t = \frac{1}{48}$ it is not (Fig. 2).

FIGURE 1. Stable approximation, $T = 1, \triangle t = \frac{1}{54}$.



Next, we consider the case of $f = 0, T = 8$ to test the asymptotic stability. If $\triangle t < 1/54$, then we should get $u^{n+1} + u^{n-1} \to 0$. The theorem fails to indicate whether $u^n \to 0$ thus we test this distinction in the Table 1.

FIGURE 2. Unstable approximation, $T = 1, \triangle t = \frac{1}{48}$.



	$\triangle t$	$\|u^N\|$	$\|u^N + u^{N-2}\|$
Test1	$\frac{1}{54}$	$2e - 12$	$5.8e - 12$
Test2	$\frac{1}{48}$	$3.1e + 15$	$4.6e + 12$
Test3	$\frac{1}{50}$	$1.3e - 11$	$1.1e - 12$

Table 1

In Test 2, $\triangle t > 1/54 > 1/100$ and $\|u^n\| \nrightarrow 0$, consistent with the theory. In test 1, $1/100 < \triangle t < 1/54$ and $\|u^n + u^{n-2}\| \to 0$ confirming the theory. Further, in Test 1 we get $u^n \to 0$. In Test 3, $\triangle t > 1/54$ (slightly), while $\|u^n\| \to 0$. Test 3 suggests that the time step limit is reasonably sharp and that the result that $u^{n+1} + u^{n-1} \to 0$ may be improvable to $u^n \to 0$. In the figures below, we also present the plots of $\|u^n\|$ vs. time step t_n consistent with behavior seen in the tables.

4. Conclusions

We have studied two step, first order uncoupling methods. The stability analysis presented herein shows that for systems the different methods can be combined as suggested by their individual components. This result validates (low order) implicit-explicit based methods for general couplings. The numerical tests identify an open question: Whether the second time step restriction (2.2) suffices to control the asymptotic behavior of the unstable mode of LF for the combined method.

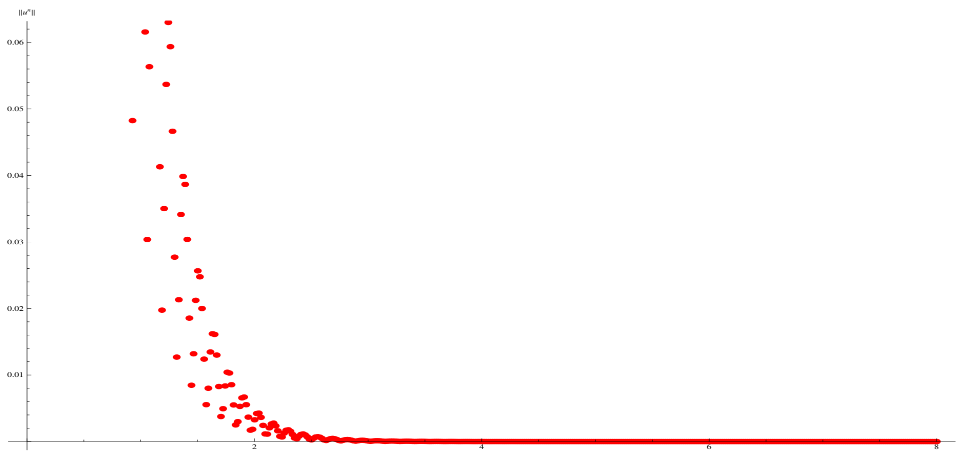


FIGURE 3. Test 1, $\triangle t = \frac{1}{54}, f = 0$

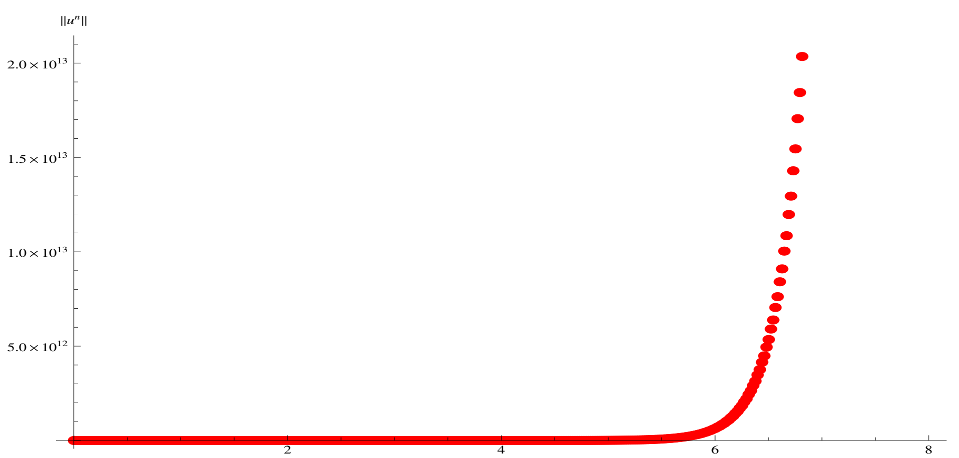


FIGURE 4. Test 2, $\triangle t = \frac{1}{48}, f = 0$

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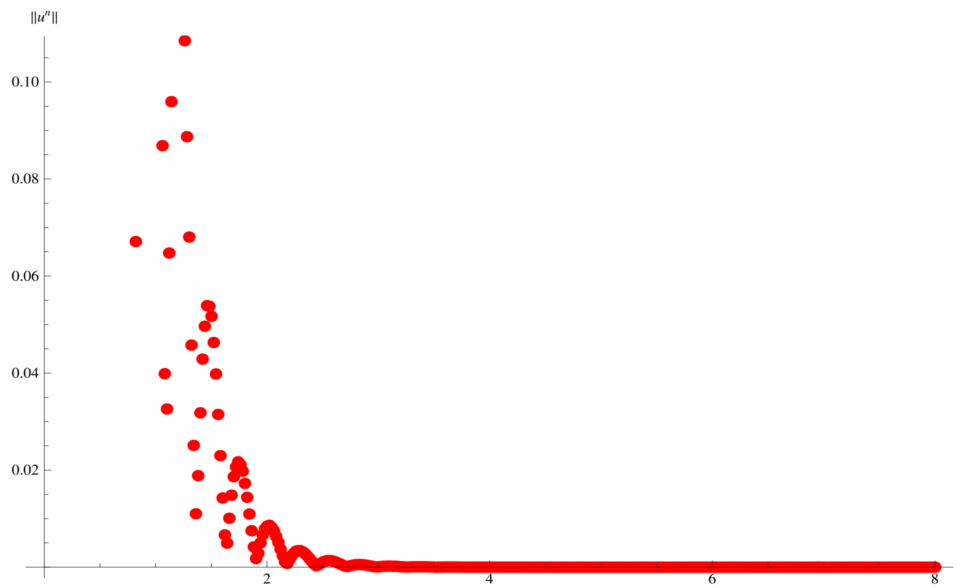

FIGURE 5. Test 3, $\triangle t = \frac{1}{50}, f = 0$

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