PRELIMINARY EXAMINATION IN LINEAR ALGEBRA
MAY 3, 2019

You may use facts *proved* in class or in Hoffman/Kunze (provide a proper reference). Justify all other assertions, including those from homework or test problems.

Problem 1.

(a) Prove or give a counterexample: if $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation such that $\operatorname{null}(T) \cap \operatorname{range}(T)$ has dimension at least 1 then T is nilpotent.

(b) Prove or give a counterexample: if $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ is a linear transformation such that $\operatorname{null}(T) \cap \operatorname{range}(T)$ has dimension at least 2 then T is nilpotent.

Problem 2. Suppose that λ is an eigenvalue of a matrix $A \in \mathbb{C}^{n\times n}$ with algebraic multiplicity k. Show that $(A - \lambda I)^k$ has rank $n - k$.

Problem 3. For any $n \geq 1$, classify the matrices $Q \in \mathbb{R}^{n\times n}$ that are both orthogonal and skew-symmetric, meaning $Q^t = -Q$, up to similarity; i.e. exhibit exactly one representative from each real similarity class. (*Hint*: the answer is very different for odd versus even n.)

Problem 4.

(a) For a diagonalizable $n \times n$ matrix A, show that $\det(e^A) = e^{\operatorname{trace}(A)}$, where e^A is the matrix exponential of A: $e^A = \displaystyle\sum_{k=0}^{\infty} \frac{1}{k!} A^k$.

(b) Now for an arbitrary 2×2 matrix A with trace equal to 0, show that $\det(e^A) = 1$. (Do not ignore the non-diagonalizable case!)

Problem 5.

(a) Show that if the self-adjoint part of a matrix A is positive-definite then A is invertible and the self-adjoint part of A^{-1} is positive-definite.

(b) Let a be a fixed positive real number. Show that if a self-adjoint matrix A is positive-definite then $\|W\| < 1$, where $W = (I - aA)(I + aA)^{-1}$ and $\|\cdot\|$ is the operator norm induced by the standard Euclidean norm.

Problem 6. Consider

$$B = \begin{pmatrix} L & M \\ O & N \end{pmatrix} \in \mathbb{C}^{2n\times 2n}$$

for $L, M, N, O \in \mathbb{C}^{n\times n}$ such that O is the zero matrix.

(a) Show that if B is diagonalizable, then L and M must be diagonalizable.

(b) Show that if L and M are diagonalizable and do not share eigenvalues, then B is diagonalizable.

Problem 1.

(a) Prove or give a counterexample: if $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation such that $\text{null}(T) \cap \text{range}(T)$ has dimension at least 1 then T is nilpotent.

(b) Prove or give a counterexample: if $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ is a linear transformation such that $\text{null}(T) \cap \text{range}(T)$ has dimension at least 2 then T is nilpotent.

Solution.

(a) Counterexample: let T have the matrix below relative to the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Note that $\mathbf{e}_1$ is an eigenvector of T with corresponding eigenvalue 1, so T is not nilpotent. On the other hand, $T\mathbf{e}_2 = \mathbf{0}$ and $T\mathbf{e}_3 = \mathbf{e}_2$, so $\mathbf{e}_2 \in \text{null}(T) \cap \text{range}(T)$.

(b) This is true. If $\text{null}(T) \cap \text{range}(T)$ has dimension at least 2 then both $\text{null}(T)$ and $\text{range}(T)$ have dimension *equal to* 2, since their dimensions add to 4 by the rank-nullity theorem and are each at least two by the hypothesis. But this implies that

$$\text{null}(T) = \text{null}(T) \cap \text{range}(T) = \text{range}(T),$$

and it follows that $T^2 = 0$.

Problem 2. Suppose that λ is an eigenvalue of a matrix $A \in \mathbb{C}^{n\times n}$ with algebraic multiplicity k. Show that $(A - \lambda I)^k$ has rank $n - k$.

Solution. Let $P \in \mathbb{C}^{n\times n}$ be an invertible matrix such that $P^{-1}AP$ is in Jordan form. Noting that

$$(P^{-1}AP - \lambda I)^k = P^{-1}(A - \lambda I)^k P$$

has the same rank as $(A - \lambda I)^k$, we may assume that A itself is in Jordan form: block diagonal with blocks $B_1, \ldots, B_m$ corresponding to the distinct eigenvalues $\lambda_1, \ldots, \lambda_m$ of A such that for each i, B_i is upper triangular with all diagonal entries equal to λ_i, $(B_i)_{j\,j+1} = 1$ or 0 for each j, and $(B_i)_{jl} = 0$ otherwise. Let us assume that $\lambda_1 = \lambda$, so that $B_1 \in \mathbb{C}^{k\times k}$.

Since I is diagonal, $A - \lambda I$ remains block-diagonal with blocks $B_i - \lambda I$ for $1 \leq i \leq m$. The diagonal entries of $B_i - \lambda I$ are $\lambda_i - \lambda$, equal to 0 if and only if $i = 1$ since $\lambda_1 = \lambda$. The blocks of $(A - \lambda I)^k$ are the $(B_i - \lambda I)^k$, each with all diagonal entries $(\lambda_i - \lambda)^k$, again equal to 0 if and only if $i = 1$. For each $i > 1$, $(B_i - \lambda I)^k$ thus has full rank, so the desired result follows from the fact that $(B_1 - \lambda I)^k$ is the zero-matrix. This in turn follows from the Cayley-Hamilton theorem, since all diagonal entries (hence eigenvalues) of $B_1 - \lambda I$ are equal to 0.

Problem 3. For any $n \geq 1$, classify the matrices $Q \in \mathbb{R}^{n\times n}$ that are both orthogonal and skew-symmetric, meaning $Q^t = -Q$, up to similarity; i.e. exhibit exactly one representative from each real similarity class. (*Hint*: the answer is very different for odd versus even n.)

Solution. Since Q is orthogonal we have

$$I = QQ^t = Q(-Q) = -Q^2,$$

so Q satisfies $m = x^2+1 \in \mathbb{R}[x]$. Since m is irreducible over $\mathbb{R}$, it is the minimal polynomial of Q, and since every eigenvalue of Q is a root of its minimal polynomial it follows that Q has no real eigenvalues. But if n is odd then since the characteristic polynomial of Q has odd degree it has a real root. It follows that there is no orthogonal, skew-symmetric matrix in $\mathbb{R}^{n\times n}$ for odd n.

Now suppose $n = 2k$ is even. The block-diagonal matrix Q with k 2×2 blocks, each of the form

$$Q_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

is skew-symmetric and orthogonal. We claim that every skew-symmetric, orthogonal matrix P is similar to Q.

"Real" Proof of the Claim. For any $\mathbf{u}_1 \in \mathbb{R}^n - \{\mathbf{0}\}$, let $\mathbf{v}_1 = P\mathbf{u}_1$. Then since $P^2 = -I$, $P\mathbf{v}_1 = -\mathbf{u}_1$. Note that neither of $\mathbf{u}_1, \mathbf{v}_1$ is a scalar multiple of the other, since P has no real eigenvalues, so $V_1 = \operatorname{span}\{\mathbf{u}_1, \mathbf{v}_1\}$ is two-dimensional and P-invariant, and Q_0 is the matrix of P on V_1 relative to the basis $\{\mathbf{u}_1, \mathbf{v}_1\}$.

We may now appeal to induction on k (where, again, $n = 2k$). The base case follows directly from above. For the inductive step, note that since P is orthogonal, it preserves the orthogonal complement $V_1^\perp$ of V_1 in $\mathbb{R}^n$. Its restriction to this $2(k-1)$-dimensional subspace remains orthogonal and skew-symmetric (as can be checked by changing to an orthogonal basis $\{\mathbf{u}_1, \mathbf{v}_1, \mathbf{w}_1, \dots, \mathbf{w}_{n-2}\}$ of $\mathbb{R}^n$, where $\{\mathbf{w}_1, \dots, \mathbf{w}_{n-2}\} \subset V_1^\perp$), so applying the inductive hypothesis we obtain a basis $\{\mathbf{u}_2, \mathbf{v}_2, \dots, \mathbf{u}_k, \mathbf{v}_k\}$ relative to which the restriction of P is block-diagonal with blocks of the form Q_0. □

"Complex" (alternative) Proof. Since P is real and orthogonal it is unitary (considered now as a complex matrix) and hence normal: $PP^* = I = P^*P$ so by the spectral theorem it is unitarily diagonalizable over $\mathbb{C}$. Because P is real and has only $\pm i$ as eigenvalues, the eigenvectors come in complex-conjugate pairs: eg. if $P\mathbf{z} = i\mathbf{z}$ then

$$P\bar{\mathbf{z}} = \overline{P\mathbf{z}} = -i\bar{\mathbf{z}}.$$

Hence there are n pairwise orthogonal unit eigenvectors $\mathbf{z}_1, \dots, \mathbf{z}_n$ of P corresponding to the eigenvalue i, and for each j the complex conjugate $\bar{\mathbf{z}}_j$ of $\mathbf{z}_j$ is an eigenvector corresponding to $-i$.

Taking $\mathbf{z}_j = \mathbf{x}_j + i\mathbf{y}_j$ for $\mathbf{x}_j$ and $\mathbf{y}_j$ real, the fact that $\mathbf{z}_j$ is orthogonal to $\bar{\mathbf{z}}_j$ gives:

$$(\mathbf{x}_j + i\mathbf{y}_j \mid \mathbf{x}_j - i\mathbf{y}_j) = \|\mathbf{x}_j\|^2 + i[(\mathbf{y}_j \mid \mathbf{x}_j) + (\mathbf{x}_j \mid \mathbf{y}_j)] - \|\mathbf{y}_j\|^2 = 0.$$

SInce $\mathbf{x}_j$ and $\mathbf{y}_j$ are real, the above thus implies that they are orthogonal and have the same norm, hence are linearly independent. And since $\mathbf{z}_j$ has corresponding eigenvalue i we have

$$P\mathbf{z}_j = P\mathbf{x}_j + iP\mathbf{y}_j = i(\mathbf{x}_j + i\mathbf{y}_j) = -\mathbf{y}_j + i\mathbf{x}_j,$$

so $P\mathbf{x}_j = -\mathbf{y}_j$ and $P\mathbf{y}_j = \mathbf{x}_j$. Therefore the real vector space $\operatorname{span}\{\mathbf{x}_j, \mathbf{y}_j\}$ is P-invariant, and the restriction of P to this subspace has matrix Q_0 relative to the basis $\{\mathbf{x}_j, \mathbf{y}_j\}$.

The proof is finished by observing that for $k \neq j$, $\mathbf{x}_k$ and $\mathbf{y}_k$ are orthogonal to each of $\mathbf{x}_j$ and $\mathbf{y}_j$ (again where $\mathbf{z}_k = \mathbf{x}_k + i\mathbf{y}_k$). This follows from using the equalities $(\mathbf{z}_j \mid \mathbf{z}_k) = 0$ and $(\bar{\mathbf{z}}_j \mid \mathbf{z}_k) = 0$ as above. □

Problem 4.

(a) For a diagonalizable $n \times n$ matrix A, show that $\det(e^A) = e^{\operatorname{trace}(A)}$, where e^A is the matrix exponential of A: $e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$.

(b) Now for an arbitrary 2×2 matrix A with trace equal to 0, show that $\det(e^A) = 1$. (Do not ignore the non-diagonalizable case!)

Solution.

(a) For an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$ we have:

$$e^A = e^{PDP^{-1}} = Pe^D P^{-1} = P \begin{pmatrix} e^{d_{11}} & 0 & \cdots & 0 \\ 0 & e^{d_{22}} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & e^{d_{nn}} \end{pmatrix} P^{-1},$$

where $d_{11}, \ldots, d_{nn}$ are the diagonal entries of D. (The second two equalities above are standard facts about the matrix exponential.) Therefore

$$\det(e^A) = \det(e^D) = e^{d_{11}} e^{d_{22}} \cdots e^{d_{nn}} = e^{\operatorname{trace}(D)}.$$

But $\operatorname{trace}(D) = \operatorname{trace}(A)$ since the trace is similarity-invariant.

(b) For a 2×2 matrix A with trace 0, we claim that if A is not diagonalizable then it is nilpotent with $A^2 = 0$. The trace of A is the sum of its eigenvalues, i.e. the roots λ_1 and λ_2 of its characteristic polynomial. Since the trace is 0 we have $\lambda_2 = -\lambda_1$. If $\lambda_1 \neq 0$ then since A has two distinct eigenvalues it is diagonalizable. So if A is not diagonalizable then $\lambda_1 = \lambda_2 = 0$. In this case the Jordan form of A is either

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

hence A is nilpotent with $A^2 = 0$ or the zero-matrix (which is diagonal).

Given the claim we need only consider the case of A non-zero with $A^2 = 0$. In this case $e^A = I + A$ satisfies the polynomial $m = (x-1)^2$, so the minimal polynomial of A divides m, and 1 is the only eigenvalue of e^A. Therefore $\det(e^A) = 1$.

Problem 5.

(a) Show that if the self-adjoint part of a matrix A is positive-definite then A is invertible and the self-adjoint part of A^{-1} is positive-definite.

(b) Let a be a fixed positive real number. Show that if a self-adjoint matrix A is positive-definite then $\|W\| < 1$, where $W = (I - aA)(I + aA)^{-1}$ and $\|\cdot\|$ is the operator norm induced by the standard Euclidean norm.

Solution.

(a) Write $A = S + T$, where S is self-adjoint and T is skew-adjoint, i.e. $T^* = -T$. Note that since T is skew-adjoint, for any $\mathbf{v}$ we have

$$(T\mathbf{v} \,|\, \mathbf{v}) = (\mathbf{v} \,|\, T^*\mathbf{v}) = -(\mathbf{v} \,|\, T\mathbf{v}) = -\overline{(T\mathbf{v} \,|\, \mathbf{v})}$$

is purely imaginary. Thus for $\mathbf{v} \neq \mathbf{0}$, $(A\mathbf{v} \,|\, \mathbf{v}) = (S\mathbf{v} \,|\, \mathbf{v}) + (T\mathbf{v} \,|\, \mathbf{v})$ is non-zero as it has positive real part. It follows that $A\mathbf{v} \neq \mathbf{0}$, hence, since $\mathbf{v} \neq \mathbf{0}$ was arbitrary, that A is non-singular.

Now for an arbitrary $\mathbf{w} \neq \mathbf{0}$, writing $\mathbf{w} = A\mathbf{v}$ for some $\mathbf{v} \neq \mathbf{0}$ we find that

$$(A^{-1}\mathbf{w} \,|\, \mathbf{w}) = (\mathbf{v} \,|\, A\mathbf{v}) = \overline{(A\mathbf{v} \,|\, \mathbf{v})}$$

has positive real part, and as above this equals $(S'\mathbf{w} \,|\, \mathbf{w})$, where S' is the self-adjoint part of A^{-1}.

(b) Since A is self-adjoint there is an orthonormal basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ consisting of eigenvectors of A. For each i let $A\mathbf{e}_i = \lambda_i \mathbf{e}_i$ for $\lambda_i > 0$. Then

$$(I - aA)\mathbf{e}_i = (1 - a\lambda_i)\mathbf{e}_i \quad \text{and} \quad (I + aA)\mathbf{e}_i = (1 + a\lambda_i)\mathbf{e}_i$$

for each i, so $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is an orthogonal basis of eigenvectors for W, with corresponding eigenvalues

$$\frac{1 - a\lambda_i}{1 + a\lambda_i}.$$

Since W is normal its operator norm is the modulus (in this case, absolute value) of its largest eigenvalue. (This can easily be proved directly.) But since a and the λ_i are both positive,

$$\frac{1 - a\lambda_i}{1 + a\lambda_i} = 1 - \frac{2a\lambda_i}{1 + a\lambda_i}$$

lies between -1 and 1 for each i.

Problem 6. Consider
$$B = \begin{pmatrix} L & M \\ O & N \end{pmatrix} \in \mathbb{C}^{2n\times 2n}$$
for $L, M, N, O \in \mathbb{C}^{n\times n}$ such that O is the zero matrix.

(a) Show that if B is diagonalizable, then L and M must be diagonalizable.

(b) Show that if L and M are diagonalizable and do not share eigenvalues, then B is diagonalizable.

Solution.

(a) If $\mathbf{w} = (\mathbf{u}, \mathbf{v})$ is an eigenvector of B with eigenvalue λ then
$$B\mathbf{w} = \begin{pmatrix} L & M \\ O & N \end{pmatrix}\begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} L\mathbf{u} + M\mathbf{v} \\ N\mathbf{v} \end{pmatrix} = \begin{pmatrix} \lambda\mathbf{u} \\ \lambda\mathbf{v} \end{pmatrix}.$$
In particular, if $\mathbf{v} \neq \mathbf{0}$ then $\mathbf{v}$ is an eigenvector of N with eigenvalue λ. For a basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_{2n}\}$ of $\mathbb{C}^{2n}$ consisting of eigenvectors for B, the second factors $\{\mathbf{v}_1, \ldots, \mathbf{v}_{2n}\}$ comprise a spanning set for $\mathbb{C}^n$, since they are the image of a spanning set under a surjective homomorphism. This set contains a basis $\{\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_n}\}$, each an eigenvector of N, and hence N is diagonalizable.

Proof 1 for L

For any $\mathbf{w}_k = (\mathbf{u}_k, \mathbf{v}_k)$ such that $\mathbf{v}_k \neq \mathbf{0}$ and $k \neq i_j$ for any j, if $B\mathbf{w}_k = \lambda_k\mathbf{w}_k$ then as above the second factor $\mathbf{v}_k$ lies in the λ_k-eigenspace of N. Thus writing $\mathbf{v}_k = \sum_{j=1}^n a_j\mathbf{v}_{i_j}$ as a linear combination of the $\mathbf{v}_{i_j}$, we have $a_j \neq 0$ only if the eigenvalue corresponding to $\mathbf{v}_{i_j}$ remains λ_k. Then $\mathbf{w}'_k = \mathbf{w}_k - \sum_{j=1}^n a_j\mathbf{w}_{i_j}$ has the property that $B\mathbf{w}'_k = \lambda_k\mathbf{w}'_k$ and that $\mathbf{w}'_k = (\mathbf{u}'_k, \mathbf{0})$ for $\mathbf{u}'_k = \mathbf{u}_k - \sum_{j=1}^n a_j\mathbf{u}_{i_j}$. Then $\mathbf{u}'_k$ is an eigenvector of L with eigenvalue λ_k. Replacing $\mathbf{w}_k$ by $\mathbf{w}'_k$ whenever $\mathbf{v}_k \neq \mathbf{0}$ and $k \neq i_j$, we obtain a new basis of $\mathbb{C}^{2n}$ with n vectors of the form $(\mathbf{u}_k, \mathbf{0})$ for eigenvectors $\mathbf{u}_k$ of L.

Proof 2 for L

To show that if B diagonalizable then L is as well, we observe that B is diagonalizable if and only if its transpose
$$B^t = \begin{pmatrix} L^t & O \\ M^t & N^t \end{pmatrix}$$
is (as one can easily show directly). For an eigenvector $\mathbf{w} = (\mathbf{u}, \mathbf{v})$ of B^t, arguing as above shows that $\mathbf{u}$ is an eigenvector for L^t and hence that the first factors of a basis of $\mathbb{C}^{2n}$ consisting of eigenvectors for B^t yield a basis of $\mathbb{C}^n$ consisting of eigenvectors for L^t, whence L^t is diagonalizable so L is as well.

(b) We will try to diagonalize B with an invertible matrix of the form:
$$\begin{pmatrix} P & K \\ O & Q \end{pmatrix}$$
For this to be invertible, P and Q must in particular be nonsingular (checking vectors of the form $(\mathbf{v}, \mathbf{0})$ and $(\mathbf{0}, \mathbf{v})$, respectively, for $\mathbf{v} \in \mathbb{C}^n - \{\mathbf{0}\}$), and in this case we have:
$$\begin{pmatrix} P & K \\ O & Q \end{pmatrix}^{-1} = \begin{pmatrix} P^{-1} & -P^{-1}KQ^{-1} \\ O & Q^{-1} \end{pmatrix}.$$

Hence,

$$\begin{aligned}
&\begin{pmatrix} P & K \\ O & Q \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ O & N \end{pmatrix} \begin{pmatrix} P & K \\ O & Q \end{pmatrix} \\
&= \begin{pmatrix} P^{-1} & -P^{-1}KQ^{-1} \\ O & Q^{-1} \end{pmatrix} \begin{pmatrix} LP & LK+MQ \\ O & NQ \end{pmatrix} \\
&= \begin{pmatrix} P^{-1}LP & P^{-1}LK + P^{-1}MQ - P^{-1}KQ^{-1}NQ \\ O & Q^{-1}NQ \end{pmatrix}
\end{aligned}$$

This matrix is diagonal if and only if both $P^{-1}LP$ and $Q^{-1}NQ$ are diagonal, and $LK+MQ-KQ^{-1}NQ = O$. Thus if L and N are diagonalizable then upon choosing P and Q so that $P^{-1}LP$ and $Q^{-1}LQ$ are diagonal, in order to diagonalize B we must solve the equation

$$MQ = KQ^{-1}NQ - LK$$

for K. We solve this equation column by column. For the jth column $\mathbf{c}_j$ of K it is equivalent to the equation $(d_jI - L)\mathbf{c}_j = (MQ)_j$, where d_j is the jth diagonal entry of the diagonal matrix $Q^{-1}NQ$ and $(MQ)_j$ is the jth column of MQ. Since d_j is an eigenvalue of N it is not an eigenvalue of L, so the matrix $d_jI - L$ is invertible and we may take $\mathbf{c}_j = (d_jI - L)^{-1}(MQ)_j$ to solve the equation.