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Instability of Crank-Nicolson Leap-Frog for Nonautonomous Systems

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The implicit-explicit combination of Crank-Nicolson and Leap-Frog methods is widely used for atmosphere, ocean and climate simulations. Its stability under a CFL condition in the autonomous case was proven by Fourier methods in 1962 and by energy methods for systems in 2012. We prove weak but unconditional instability for Leap-Frog in the nonautonomous case herein.

Keywords: CNLF; nonautonomous system; stability.

1. Introduction

This note considers stability of CNLF, the Crank-Nicolson Leap-Frog method (CNLF) below, for systems with nonautonomous $A(t),\Lambda(t)$:

$$\frac{du}{dt}+A(t)u+\Lambda(t)u=0, \text{ for } t>0, \text{ and } u(0)=u_0. \tag{1.1}$$

$A(t),\Lambda(t)$ are $d\times d$ matrices and $u(t)$ is a d vector. $A(t)$ is positive semi-definite symmetric part and $\Lambda(t)$ is skew symmetric. Let $|\cdot|_2$ denote the euclidean norm. Under the timestep condition

$$\Delta t|\Lambda|_2 \leqslant \alpha < 1, \tag{1.2}$$

stability in the autonomous, scalar case was proven in 1963 in Johansson and Kreiss (9), see also (4), and for non-commuting, autonomous systems in 2012 (12), see also (17) for background. We prove herein weak instability in the nonautonomous case.

The extension of stability for ODEs from autonomous to nonautonomous (with test problem $y'=\lambda(t)y$) has a rich history. Dahlquist (3) proved that an A-stable method is similarly stable for $y'=\lambda(t)y$ when $Re(\lambda(t))\leqslant 0$, further developed in (13). For the corresponding AN-stability theory for Runge-Kutta methods, see Hundsdorfer and Stetter (7). For non-A-stable multi-step methods, nonautonomous stability theory was recently developed in Boutelje and Hill (2). Their theory gives conditions under which a method will be stable for $y'=\lambda(t)y$ and under which it will be unstable. For example, given a linear multistep method for $y'=\lambda(t)y$, let $\rho(z),\sigma(z)$ be the complex polynomials associated with the method in a standard way and form

$$a(i):=Re\left[\left.\frac{\rho(z)}{\sigma(z)}\right|_{z=i}\right].$$

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If Δt small enough to be in the stability region of the method, and $a(i) < 0$ then there exists a $\lambda(t) < 0$ for which the method is unstable, (2).

In the Boutelje-Hill theory, the leap-frog method (CNLF with $A(t) = 0$) is an important wedge example. Indeed, for leap frog, we calculate $\rho(z) = \frac{1}{2}z^2 - \frac{1}{2}$, and $\sigma(z) = z$. Thus $a(i) \equiv 0$. Many interesting behaviors are possible between exponential asymptotic stability and exponential instability. One hint is that there is a rich catalog (e.g., (1),(14),(15),(18)) of exotic behavior of leap-frog for Burgers equation starting (to our knowledge) with Fornberg's 1973 paper (6). CNLF is also particularly important since it is the method used for the dynamic core of most current atmosphere, ocean and climate codes, e.g., (4), (10), (16), and other geophysics problems, (11) .

We consider thus CNLF: let $t^n = n\Delta t$; given u^0, u^1 find $u^n \in X$ for $n \geqslant 2$ satisfying

$$\frac{u^{n+1} - u^{n-1}}{2\Delta t} + A(t^n)\frac{u^{n+1} + u^{n-1}}{2} + \Lambda(t^n)u^n = 0, \tag{CNLF}$$

with approximations to appropriate accuracy, (17), at the first two time steps.

To summarize the results, first suppose

$$|\Lambda(t^n) - \Lambda(t^{n-1})|_2 \leqslant a_0 \Delta t. \tag{1.3}$$

We prove in Theorem 1 the upper bound

$$|u^{N+1}|_2^2 + |u^N|_2^2 \leqslant C(\alpha, u^0, u^1) \exp\left[\Delta t \frac{a_0}{1-\alpha} t^N\right]. \tag{1.4}$$

For Lipschitz $\Lambda(t)$, the rate constant $\Delta t a_0 \to 0$ as $\Delta t \to 0$ but $\exp\left[\frac{\Delta t a_0 t^N}{1-\alpha}\right] \to \infty$ as $t^N \to \infty$. However, the true solution $u(t) \to 0$ as $t \to \infty$. Thus, if this estimate is sharp, CNLF has a weak instability in the nonautonomous case. This raises the possibilities that either (i) the bound, while unusual, may be sharp, (ii) the true rate of growth may be linear (or polynomial) in t^N, or (iii) stability may require both (1.2) and a second timestep condition. We give two constructions that show that (1.4) is best possible.

EXAMPLE 1.1 (Exponential instability under (1.2)) Take $A(t) = 0$ (so CNLF reduces to LF), $\Delta t = 1/2$, $u = (u_1, u_2)^t$, $y^0 = (1,1)^t$, $y^1 = (1,-1)^t$ and choose $\ell(t) = \cos(2\pi t)$. Consider LF for the 2×2 system:

$$u' + \Lambda(t)u = 0, \Lambda(t) = \ell(t)\begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix}. \tag{1.5}$$

In this example

$$\Delta t|\Lambda(t^n)|_2 = \frac{1}{2} < 1 \ \text{ but } \ \Delta t|\Lambda(t^n) - \Lambda(t^{n-1})|_2 = 1.$$

LF is given by: u^0, u^1 given, $u^{n+1} = u^{n-1} - 2\Delta t \Lambda(t^n)u^n$, where

$$\Lambda(t^n) = (-1)^{n+1}\begin{bmatrix} 0 & +1 \\ -1 & 0 \end{bmatrix}$$

It is easily proven by induction that u^n is given by $u^n = F_n(1, (-1)^{n+1})^T$ where $\{F_n\}$ is the Fibonacci sequence. Thus $|u^n| \to \infty$, as $n \to \infty$.

The instability in this example is so dramatic that it is surprising that in our numerical tests stability seemed to be the generic behavior.

EXAMPLE 1.2 (For smaller Δt the instability appears to vanish) We computed the growth rates of CNLF for smaller time steps in the previous example with $\ell(t^n) = \cos(2\pi t^n)$, and (smaller) timesteps $\Delta t = P/K$ for integers K, $P \leqslant K$, $K \leqslant 1400$. We applied LF, (CNLF) with A(t)=0.

The average growth rate ρ was computed by constructing the 2×2 matrix given in (3.6) below and computing its eigenvalues λ_j. The value $\sigma = \max\{|\lambda_j|\}$ is the largest possible net growth attained for some starting values after K steps of LF. Thus, for some starting values, the n^{th} LF step satisfies $|u^n| \approx |u^0|\sigma^{n/K} = |u^0|e^{\rho n}$, with $\rho = \frac{1}{K}\log\sigma$. When $\rho = 0$, (we shall take 10^{-8} as numerically zero) the iteration is stable while if $\rho > 0$ the LF iterates grow exponentially.

Figure 1 plots the largest growth rate, ρ, attainable (by suitable choice of starting value) of LF *vs.* Δt. The vertical axis is ρ scaled logarithmically and labelled with ρ values and the horizontal axis is Δt. Values of ρ smaller than 10^{-8} can be regarded as numerically zero because it would take 10^6 steps before u^n would increase 1% in size.

Notice that the values of Δt which give $\rho = 0$ (to numerical precision) are overwhelmingly more common than values of Δt for which ρ is of significant size. Thus, a randomly-chosen Δt along with randomly-chosen initial values would be unlikely to produce an instability. Notice also that as Δt becomes smaller, values of Δt which produce an instability become even less common. It even appears that there are no values of Δt which produce an instability for $\Delta t < 0.05$. (We shall show that this is not true for a different $\ell(t)$ in section 3.)

It is hard to observe a pattern in the scattered values of Δt which yield instabilities in Figure 1. It is clear that values of Δt that give rise to significant growth rates are not easy to guess.

In Section 3 we give a construction that shows that LF (and thus CNLF) is exponentially unstable *for arbitrarily small timesteps* for (1.5) when $\ell(t)$ is a bounded function that changes sign periodically. The example in Section 3 shows conclusively that the upper estimate in Theorem 2.1 is attained.

2. CNLF when $\Lambda = \Lambda(t)$

We prove the claimed stability bound for the CNLF method.

THEOREM 2.1 Assume for every t^n (1.2) holds. Then CNLF satisfies: for every $N \geqslant 2$

$$|u^{N+1}|_2^2 + |u^N|_2^2 \leqslant C(\alpha, u^0, u^1)\exp\left[\frac{\Delta t}{1-\alpha}\sum_{n=1}^{N}|\Lambda(t^{n+1}) - \Lambda(t^n)|_2^2\right], \tag{2.1}$$

where $C(\alpha, u^0, u^1) = [\,|u^1|_2^2 + |u^0|_2^2 + 2\Delta t(u^1)^T\Lambda(t^0)u^0]/(1-\alpha)$.

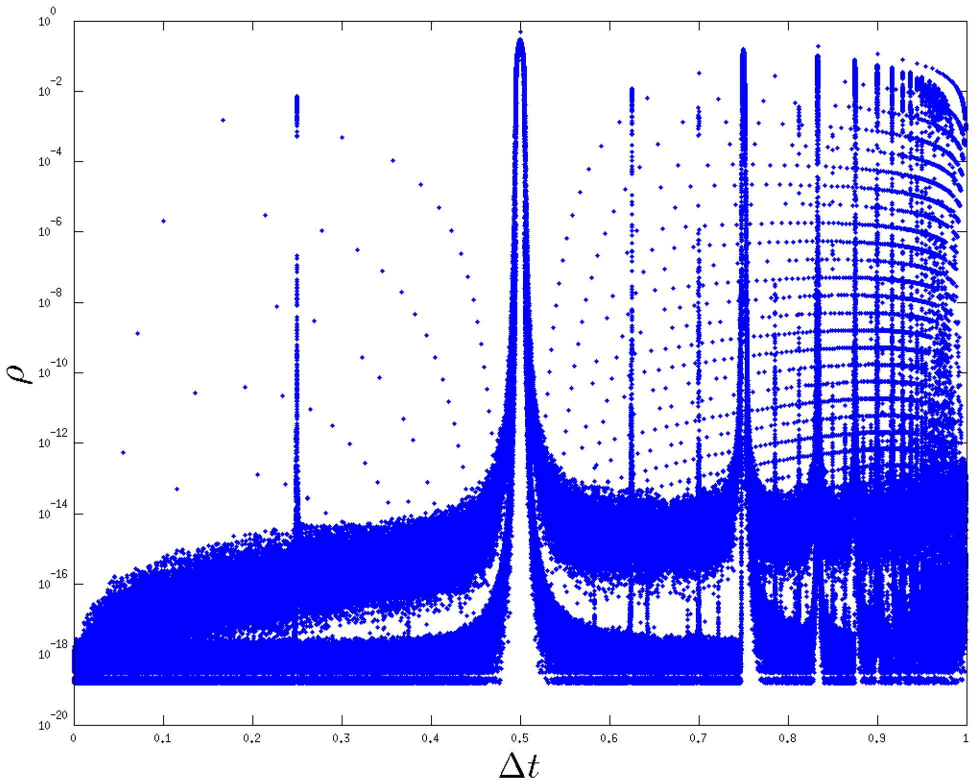
Proof. Define $E^{n+1/2} := |u^{n+1}|^2 + |u^n|^2$. Take the dot product of (CNLF) with $\left(u^{n+1} + u^{n-1}\right)$ and add and subtract $|u^n|_2^2$. This gives

$$[E^{n+1/2} - E^{n-1/2}] + \Delta t\left(u^{n+1} + u^{n-1}\right)^T A(t^n)\left(u^{n+1} + u^{n-1}\right) +$$
$$+2\Delta t\left(u^{n+1} + u^{n-1}\right)^T \Lambda(t^n)u^n = 0.$$

Since A is positive semi-definite $\left(u^{n+1} + u^{n-1}\right)^T A(t^n)\left(u^{n+1} + u^{n-1}\right) \geqslant 0$ and can be dropped. From skew symmetry of $\Lambda(t^n)$ we can write

$$[E^{n+1/2} - E^{n-1/2}] + 2\Delta t\left[\left(u^{n+1}\right)^T \Lambda(t^n)u^n - (u^n)^T \Lambda(t^n)u^{n-1}\right] \leqslant 0.$$

FIG. 1. Average growth rate ρ *vs.* Δt for $\ell(t) = \cos(2\pi t)$



Rewrite the Λ terms as $C^{n+1/2} - C^{n-1/2} - Q^n$ with Q^n the extra term that now occurs due to time dependence

$$\left(u^{n+1}\right)^T \Lambda(t^n)u^n - (u^n)^T \Lambda(t^n)u^{n-1} = C^{n+1/2} - C^{n-1/2} - Q^n,$$
$$C^{n+1/2} := \left(u^{n+1}\right)^T \Lambda(t^n)u^n, \quad C^{n-1/2} := (u^n)^T \Lambda(t^{n-1})u^{n-1},$$
$$Q^n := (u^n)^T \left[\Lambda(t^n) - \Lambda(t^{n-1})\right] u^{n-1}.$$

Note that by the usual inequalities $2Q^n \leqslant |\Lambda(t^n) - \Lambda(t^{n-1})|_2 E^{n-1/2}$, giving, for $n \geqslant 1$,

$$E^{n+1/2} + 2\Delta t C^{n+1/2} \leqslant E^{n-1/2} + 2\Delta t C^{n-1/2} + \Delta t |\Lambda(t^n) - \Lambda(t^{n-1})|_2 E^{n-1/2}$$

The timestep condition (1.2) implies $E^{n+1/2} + \Delta t C^{n+1/2} > (1-\alpha)E^{n+1/2}$. Summing we have

$$E^{N+1/2} \leqslant \frac{1}{1-\alpha}\left[E^{1/2} + \Delta t C^{1/2}\right] + \frac{\Delta t}{1-\alpha}\sum_{n=1}^{N} |\Lambda(t^n) - \Lambda(t^{n-1})|_2 E^{n-1/2}.$$

Thus by the discrete Gronwall lemma we have, for every $N \geqslant 1$

$$E^{N+1/2} \leqslant \frac{1}{1-\alpha}\left[E^{1/2}+\Delta t C^{1/2}\right]+$$
$$\left[E^{1/2}+\Delta t C^{1/2}\right]\frac{\Delta t}{(1-\alpha)^2}\sum_{n=1}^{N}|\Lambda(t^n)-\Lambda(t^{n-1})|_2\cdot M_N,$$

where $M_N := \prod_{n<j\leqslant N}\left(1+\frac{\Delta t}{1-\alpha}|\Lambda(t^j)-\Lambda(t^{j-1})|_2\right)$.

In particular, as a special case

$$E^{N+1/2} \leqslant \frac{1}{1-\alpha}\left[E^{1/2}+\Delta t C^{1/2}\right]\exp\left[\frac{\Delta t}{1-\alpha}\sum_{n=1}^{N}|\Lambda(t^n)-\Lambda(t^{n-1})|_2\right].$$

This completes the proof under the assumed timestep restriction. □

How big is the predicted growth rate? Consider the growth term on the RHS of (2.1). If $\Lambda(t)$ is Lipschitz continuous (1.3) then

$$\exp\left[\frac{\Delta t}{1-\alpha}\sum_{n=1}^{N}|\Lambda(t^n)-\Lambda(t^{n-1})|_2\right]\leqslant \exp\left[\frac{\Delta t}{1-\alpha}a_0 t^N\right].$$

Returning to the proof, the growth rate arises from

$$\left(1+\frac{a_0\Delta t^2}{1-\alpha}\right)^N \leqslant \left(1+\frac{a_0\Delta t^2}{1-\alpha}+\frac{\left(a_0\Delta t^2\right)^2}{(1-\alpha)2!}+\cdots\right)^N \leqslant \exp\left(\frac{\Delta t}{1-\alpha}a_0 N\Delta t\right),$$

in which the first step obviously is not sharp. If we rescale by $s=\frac{a_0}{1-\alpha}(n\Delta t)^2=\frac{a_0}{1-\alpha}(t^n)^2, m=n^2$, we have $\left(1+\frac{a_0}{1-\alpha}\Delta t^2\right)^n=\left(1+\frac{s}{m}\right)^{\sqrt{m}}$. Sharp double asymptotic limits $m\to\infty$ and $s\to\infty$ can be obtained using calculus giving a slight improvement for large timesteps:

$$\left(1+\frac{a_0}{1-\alpha}\Delta t^2\right)^N \leqslant \begin{cases} \exp\left(\frac{\Delta t a_0 t^N}{1-\alpha}\right), \text{ for } \Delta t<\sqrt{0.6117a_0^{-1}(1-\alpha)} \\ \exp\left(\frac{0.807a_0 t^N}{1-\alpha}\right), \text{ for } \Delta t>\sqrt{0.6117a_0^{-1}(1-\alpha)} \end{cases}$$

3. Exponential growth for small Δt

In this section, we present a construction of a 2×2 system (1.1) for which LF is unstable for arbitrarily small timesteps. Let $A(t)=0$ so CNLF reduces to LF. The skew symmetric matrix Λ is constructed to satisfy

$$|\Lambda(t)|_2 \equiv 1 \text{ and } |\Lambda(t^n)-\Lambda(t^{n-1})|_2 \leqslant 2.$$

A sequence $\Delta t^k \to 0$ is exhibited for which the solution to (CNLF) grows as $e^{\Delta t^k ct}$. This example shows that the RHS of (2.1) has asymptotic behavior that cannot be improved and thus (CNLF) is unconditionally unstable.

Choose $\ell(t)$ a periodic function (specified below) and

$$\Lambda(t)=\ell(t)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

he construction is performed in several steps:

1. Define a sequence of complex z^n derived from the original sequence v^n.

2. Rewrite the nonautonomous recursion (CNLF) as an *autonomous* recursion $A_1Z^m = A_2Z^{m-1}$. Where the complex vectors Z^m are defined by grouping the iterates z^n according to the period of $\ell(t)$.

3. Write an explicit recursive expression for $Z^m = A_1^{-1}A_2Z^{m-1}$.

4. Construct a 2×2 matrix $B_{2\times 2}$ whose eigenvalues agree with the nontrivial eigenvalues of $A_1^{-1}A_2$.

5. Construct a specific $\ell(t)$.

6. Show that the eigenvalues of $B_{2\times 2}$ are real and one of them has magnitude $1+2\Delta t+O(\Delta t^2)>1$.

Step 1. Equivalent complex recursion. Rewrite the vector (u_1,u_2) as a complex scalar $z=u_1+iu_2$. Since $A=0$, (CNLF) becomes

$$z^{k+1} = z^{k-1} - 2i\Delta t\, \ell(t^k) z^k. \tag{3.1}$$

Step 2. Autonomous recursion. Assume that $\ell(k\Delta t)$ is periodic with period $K\Delta t$. Set $a_k = 2\Delta t\, \ell(k\Delta t)$. Periodicity implies that $a_{k+K}=a_k$. Define a complex vector Z^m with components Z_k^m, for $k=1,\ldots,K$ as

$$Z_k^m = z^{mK+k}. \tag{3.2}$$

Substituting $m-1$ for m gives $Z_k^{m-1} = z^{mK+k-K}$.

From (3.1) we have

$$\begin{aligned}
z^{mK+1} &= z^{mK-1} - (ia_{mK})z^{mK}\\
z^{mK+2} &= z^{mK} - (ia_{mK+1})z^{mK+1}\\
z^{mK+3} &= z^{mK+1} - (ia_{mK+2})z^{mK+2}\\
z^{mK+4} &= z^{mK+2} - (ia_{mK+3})z^{mK+3}\\
&\vdots\\
z^{mK+K} &= z^{mK+K-2} - (ia_{mK+K-1})z^{mK+K-1}
\end{aligned}$$

Writing this in terms of Z_k^m gives

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ ia_1 & 1 & 0 & 0 & \cdots & 0 \\ -1 & ia_2 & 1 & 0 & \cdots & 0 \\ 0 & -1 & ia_3 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & -1 & ia_{K-1} & 1 \end{bmatrix} \begin{bmatrix} Z_1^m \\ Z_2^m \\ Z_3^m \\ Z_4^m \\ \vdots \\ Z_K^m \end{bmatrix} = \tag{3.3}$$

$$= \begin{bmatrix} 0 & 0 & 0 & \cdots & 1 & -ia_K \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} Z_1^{m-1} \\ Z_2^{m-1} \\ Z_3^{m-1} \\ Z_4^{m-1} \\ \vdots \\ Z_K^{m-1} \end{bmatrix}$$

Denoting the two matrices in (3.3) as A_1 and A_2,

$$A_1 Z^m = A_2 Z^{m-1} \tag{3.4}$$

where Z^m is the vector of Z_k^m and A_1 and A_2 are the above matrices.

Step 3. Explicit recursion expression. The only nontrivial columns of the matrix A_2 are the final two, so the the only nontrivial columns in the product $B = A_1^{-1} A_2$ are its final two columns. Furthermore, the matrix A_1 is lower triangular with only three nontrivial diagonals, so B can be written in recursive form as

$$\begin{aligned} B_{1,K-1} &= 1 \\ B_{2,K-1} &= -ia_K \\ B_{1,K} &= -ia_1 B_{1,K-1} \\ B_{2,K} &= 1 - ia_1 B_{2,K-1} \\ B_{k,j} &= B_{k-2,j} - ia_{k-1} B_{k-1,j} \text{ for } k = 3, \ldots, K \text{ and } j = K-1, K \end{aligned} \tag{3.5}$$

Step 4. Reduction to a 2×2. Since the recursion (3.4) is autonomous, its stability is determined by the spectral radius of the matrix B. It turns out that the spectral radius of B is equal to the spectral radius of a derived 2×2 matrix $B_{2\times 2}$. We will choose a function $\ell(t)$ for which the spectral radius of $B_{2\times 2}$ is larger than one for a sequence of values $K \to \infty$.

Because of its structure, there are necessarily $K-2$ null eigenvalues of B. If a vector Z is an eigenvector of B with eigenvalue $\lambda \neq 0$, then the following 2×2 system must be satisfied.

$$B_{2\times 2} \begin{bmatrix} Z_{K-1} \\ Z_K \end{bmatrix} = \begin{bmatrix} B_{K-1,K-1} & B_{K-1,K} \\ B_{K,K-1} & B_{K,K} \end{bmatrix} \begin{bmatrix} Z_{K-1} \\ Z_K \end{bmatrix} = \lambda \begin{bmatrix} Z_{K-1} \\ Z_K \end{bmatrix}. \tag{3.6}$$

Each eigenvector of $B_{2\times 2}$ can be expanded into an eigenvector of B, by choosing its two components as the two initial values in (3.1), so the non-null eigenvalues of B and $B_{2\times 2}$ agree.

Step 5. Choice of ℓ. Choose $\ell(t)$ to be the periodic function of period 1

$$\ell(t) = \begin{cases} +1 & 0 \leqslant t < 1/4 \\ -1 & 1/4 \leqslant t < 3/4 \\ +1 & 3/4 \leqslant t < 1 \\ \text{periodic} & \text{otherwise} \end{cases} \tag{3.7}$$

For an *even*[1] integer k_0, let $K = 2(2k_0+1)$, and $\Delta t = 1/K$. With this choice of K, the values $t^k = k\Delta t$ never exactly equal a point of discontinuity of $\ell(t)$, and Δt *can be chosen arbitrarily small.*

Step 6. Spectral radius. A straightforward but tedious induction shows that

$$B_{2\times 2} = \begin{bmatrix} 1 & -2i\Delta t \\ 2i\Delta t & 1+4\Delta t^2 \end{bmatrix} + O(\Delta t^3). \tag{3.8}$$

The eigenvalues of $B_{2\times 2}$ are

$$\lambda = 1 \pm 2\Delta t + 2\Delta t^2 + O(\Delta t^3).$$

Thus, the spectral radius is larger than 1

$$\sigma = \operatorname{spr}(B_{2\times 2}) = \operatorname{spr}(A_1^{-1}A_2) = 1 + 2\Delta t + O(\Delta t^2) > 1.$$

Choose an initial vector Z^0 as the dominant eigenvector of B, $Z^k = \sigma Z^{k-1}$. Since each vector Z^k represents $K = 1/\Delta t$ LF timesteps, the complex iterates satisfy $|z^{Kn}| = |z^0|e^{(Kn\Delta t)\log\sigma} + O(\Delta t^2)$. On average, then, with $t^n = n\Delta t$, $|z^n| \approx |z^0|e^{(2\Delta t/K)t^n} + O(\Delta t^2)$. Denote the average growth rate per timestep as $\rho = 2\Delta t + O(\Delta t^2)$.

For each *even* integer $k_0 = 2, 4, \ldots$, letting $K = 2(2k_0+1)$, and $\Delta t = 1/K \to 0$ results in an exponentially divergent LF iteration. The average rate of growth of the iterates is $\rho = 2\Delta t + O(\Delta t^2)$. There is no limiting size of Δt below which the iteration does not diverge.

This choice of $\Lambda(t)$ is not Lipschitz, but Theorem 2.1 does apply. The exponential factor in the theorem is

$$\exp\left[\frac{\Delta t}{1-\alpha}\sum_{n=1}^{N} |\Lambda(t^{n+1}) - \Lambda(t^n)|_2^2\right].$$

For this choice of Λ, the difference $|\Lambda(t^{n+1}) - \Lambda(t^n)|_2^2$ is nonzero (and then equal to 4) only twice per period. Replacing N with nK, the exponential factor becomes

$$\exp\left[\frac{\Delta t}{1-\alpha}4nK(2/K)\right] = \exp\left[\frac{\Delta t}{1-\alpha}8t\right]$$

since $nK\Delta t = t$. Thus the average growth rate in Theorem 2.1 is $\rho = 8\Delta t/(1-\alpha)$, and α can be taken to be small.

REMARK 3.1 It is interesting to note that (3.8) contains no explicit dependence on the value k_0. This is a consequence of the average of $\ell(t)$ over a period being zero. Just after the point $t = k_0/K$, where $\ell(t)$ jumps from +1 to -1, the analogous matrix is

$$\begin{bmatrix} -ik_0\Delta t & 1 - k_0(k_0+2)/2\Delta t^2 \\ 1 - k0(k0+2)/2\Delta t^2 & -i(k_0+2)\Delta t \end{bmatrix}$$

The function $\ell(t)$ was chosen because it has a Fourier series with terms related to the function in the first example, $\ell(t) = (4/\pi)\sum_{n=0}^{\infty} cos(2(2n+1)\pi t)/(2n+1)$. For $K = 2(2k_0+1)$, and $\Delta t = 1/K$, as above, the term $n = k_0$ takes the values ± 1, so that growth appears just as in the first example.

The leapfrog method for the differential equation

$$\frac{du}{dt} - i\ell(t)u = 0$$

[1]Odd integers work similarly and with the same spectral radius, but the formulæ are slightly different.

is precisely (3.1), and the solution of this equation, starting with $u(0) = u_o$ is

$$u(t) = \begin{cases} u_0 e^{it} & 0 \leqslant t < 1/4 \\ u_o e^{-i(t-1/2)} & 1/4 \leqslant t < 3/4 \\ u_0 e^{i(t-1)} & 3/4 \leqslant t < 1 \\ \text{periodic} & t > 1 \end{cases} \tag{3.9}$$

This function is bounded, so the growth of the leapfrog approximation is due to the numerical approximation.

4. Conclusions

In contrast to the autonomous case, in the nonautonomous case we have shown through two constructions that CNLF (and, as a special case, leapfrog) computed solutions can grow even when the system itself has bounded or decaying solutions. In Theorem 2.1 we prove that the growth rate, however, is at worst proportional to Δt when $\Lambda(t)$ is Lipschitz, a rate attained in the examples constructed.

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