PARTITIONED SECOND ORDER METHOD FOR MAGNETOHYDRODYNAMICS IN ELSÄSSER FIELDS

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Abstract. Magnetohydrodynamics (MHD) studies the dynamics of electrically conducting fluids, involving Navier-Stokes equations coupled with Maxwell equations via Lorentz force and Ohm's law. Monolithic methods, which solve fully coupled MHD systems, are computationally expensive. Partitioned methods, on the other hand, decouple the full system and solve subproblems in parallel, and thus reduce the computational cost.

In this report we propose and analyzes a second-order in time partitioned method for the MHD system in the Elsässer variables. We perform stability analysis, show that the method is stable for the magnetic Prandtl number of order unity, derive error estimates and present a numerical test supporting the theoretical results.

1. Introduction. Magnetohydrodynamics (MHD) studies the interaction between the electrically conducting fluids and the electromagnetic fields. Initiated by Alfvén in 1942 [1], MHD is widely exploited in numerous branches of science including astrophysics and geophysics [17, 26, 12, 9, 8, 3, 5, 11], as well as engineering. Understanding MHD flows is central to many important applications, e.g., liquid metal cooling of nuclear reactors [2, 16, 29], process metallurgy [6], MHD propulsion [22, 25].

The MHD flows entail two distinct physical processes: the motion of fluid is governed by hydrodynamics equations and the magnetic field is governed by Maxwell equations. One approach to solve the coupled problem is by monolithic methods, or fully coupled implicit algorithms (e.g., [35]). In these methods, the globally coupled problem is assembled at each time step and then solved iteratively. Thus although robust and stable, they are quite demanding in computational time and resources.

In contrast, partitioned methods are semi-implicit algorithms that treat subphysics/subdomain problems implicitly and the coupling terms explicitly. Hence such methods are able to solve subproblems in parallel and significantly reduce the computational complexity. Partitioned methods are widely used in ocean-atmosphere models, see e.g., [19]. However, there has been much less work on time-dependent MHD. To the best of our knowledge, such methods are proposed in [31], [34] and [21]. The first two papers developed unconditionally stable, first order and second order partitioned methods for full MHD based on decoupling Elsässer fields respectively, while the last paper presented such methods for reduced MHD.

In this report we propose a two step, second-order partitioned method for MHD in Elsässer fields that adopts implicit discretization of the subproblem terms and explicit discretization of coupling terms. The stability analysis shows that the method is unconditionally stable if the magnetic Prandtl number, Pr_m, satisfies $1/2 < Pr_m < 2$, and is conditionally stable otherwise. In addition, the algorithm is shown to be long-time stable in the sense that the energy is bounded uniformly in time. We also perform numerical tests to verify the theory.

To specify the problem considered, we describe the full MHD equations below. Given a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, and time $T > 0$, the fluid velocity field u, the magnetic field B (rescaled to give it dimensions of a velocity), and the total pressure p (kinetic and magnetic) satisfy [4, 27]

$$\frac{\partial u}{\partial t} + u \cdot \triangledown u - B \cdot \triangledown B - \nu \triangle u + \triangledown p = f, \qquad \triangledown \cdot u = 0, \tag{1.1}$$

$$\frac{\partial B}{\partial t} + u \cdot \triangledown B - B \cdot \triangledown u - \nu_m \triangle B = 0, \qquad \triangledown \cdot B = 0, \tag{1.2}$$

where ν is the kinematic viscosity, ν_m is the magnetic resistivity, and f is a body force.

An important dimensionless parameter in MHD is the *magnetic Prandtl number* $Pr_m := \nu/\nu_m$. In practice it may vary considerably depending on the medium, for instance $Pr_m = 7$ for water, $\simeq 0.7$ for air,

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and $\sim 10^{-5}$ in the liquid core of the Earth. In a number of laboratory simulation, however, the magnetic Prandtl number is taken to be unity, or of order unity, see e.g., [4, 15, 24, 30] and references therein.

The total magnetic field can be split in two parts, $B = B_0 + \tilde{B}$, where B_0 and $\tilde{B}$ are mean and fluctuation, respectively. The Elsässer fields [10]

$$z^+ = u + \tilde{B}, \quad z^- = u - \tilde{B},$$

merging the physical properties of the Navier-Stokes and Maxwell equations. By adding (1.1) and (1.2), or subtracting (1.2) from (1.1), we obtain the momentum equations in Elsässer fields:

$$\begin{aligned} \frac{\partial z^\pm}{\partial t} \mp (B_\circ \cdot \nabla) z^\pm + (z^\mp \cdot \nabla) z^\pm - \frac{\nu + \nu_m}{2} \Delta z^\pm - \frac{\nu - \nu_m}{2} \Delta z^\mp + \nabla p &= f, \\ \nabla \cdot z^\pm &= 0, \end{aligned} \tag{1.3}$$

The interesting property of the Elsässer fields is that there is no self-coupling in the nonlinear term in (1.3), but only cross-coupling of z^+ and z^-. This is the basis of the Alfvén effect, which describes a fundamental interaction process, see [18, 20, 7, 23, 32, 28, 13, 14, 33]. From the point of computational view, this property may suggest the use of partitioned methods.

The paper is organized as follows. Section 2 introduces notation and necessary preliminaries. We then describe the partitioned method and perform stability analysis in Section 3. The error estimate is derived in Section 4, and a numerical test is presented in Section 5. Finally, Section 6 concludes the paper.

2. Notation and preliminaries. Throughout this paper, we denote the $L^2(\Omega)$-norm by $\|\cdot\|$ and the corresponding inner product by $(\cdot,\cdot)$, and the norm in $H^k(\Omega)$ by $\|\cdot\|_k$. For functions $v(x,t)$ defined in $\Omega \times (0,T)$, we introduce the following norms

$$\|v\|_{\infty,k} := \operatorname*{ess\,sup}_{t\in[0,T]} \|v(\cdot,t)\|_k, \quad \text{and} \quad \|v\|_{m,k} := \left(\int_0^T \|v(\cdot,t)\|_k^m \right)^{1/m},$$

as well as the discrete norms

$$|||v|||_{\infty,k} := \max_{0\le n\le T/\Delta t} \|v_n\|_k, \quad |||v|||_{2,k} := \left(\Delta t \sum_{n=0}^{T/\Delta t} \|v_n\|_k^2 \right)^{1/2},$$

where Δt denotes the time step.

Recall that the G-norm of a function $\mathbf{w} = (w_1, w_2)^T \in \left(L^2(\Omega)\right)^2$ is defined by

$$\|\mathbf{w}\|_G^2 := (\mathbf{w}, G\mathbf{w}),$$

where G is a 2×2, symmetric positive definite matrix. The specific G matrix in our problem is given by

$$G = \begin{pmatrix} \frac{5}{2} & -1 \\ -1 & \frac{1}{2} \end{pmatrix},$$

and consequently, $\|\mathbf{w}\|_G^2 = \frac{1}{2}\left(\|w_1\|^2 + \|2w_1 - w_2\|^2\right)$. Introducing the *central difference operator* $D_2 v_{n+1} = v_{n+1} - 2v_n + v_{n-1}$, and letting $\mathbf{w_n} = (v_n, v_{n-1})^T$, we have the following identity:

$$(3v_{n+1} - 4v_n + v_{n-1}, v_{n+1}) = \|\mathbf{w_{n+1}}\|_G^2 - \|\mathbf{w_n}\|_G^2 + \frac{1}{2}\|D_2 v_{n+1}\|^2.$$

Note that the G-norm is equivalent to norm on $\left(L^2(\Omega)\right)^2$ in the sense that

$$\left(\frac{3-2\sqrt{2}}{2}\right)\|\mathbf{w}\| \le \|\mathbf{w}\|_G \le \left(\frac{3+2\sqrt{2}}{2}\right)\|\mathbf{w}\|.$$

The spaces of Elsässer fields and pressure are defined by, respectively,

$$X = \left(H_0^1(\Omega)\right)^d = \left\{v \in \left(L^2(\Omega)\right)^d, \nabla v \in \left(L^2(\Omega)\right)^{d\times d},\ v = 0 \text{ on } \partial\Omega\right\},$$
$$Q = L_0^2(\Omega) = \left\{q \in L^2(\Omega), \int q dx = 0\right\},$$

and the divergence-free function space is

$$V = \{v \in X : (\bigtriangledown \cdot v, q) = 0,\ \forall q \in Q\}.$$

Define the bilinear form $a(\cdot,\cdot) : X \times X \to \mathbb{R}$,

$$a(u, v) := (\nabla u, \nabla v),$$

which is continuous and coercive. Due to the divergence-free condition, we write the nonlinear term $(u \cdot \nabla v, w)$ in a trilinear form, $b(\cdot,\cdot,\cdot) : X \times X \times X \to \mathbb{R}$,

$$b(u, v, w) := \frac{1}{2}\big[(u \cdot \bigtriangledown v, w) - (u \cdot \bigtriangledown w, v)\big].$$

Note that there exists a generic constant $C = C(\Omega)$ such that []

$$\begin{aligned} |b(u, v, w)| &\le C\|\nabla u\|\|\nabla v\|\|\nabla w\|, \\ |b(u, v, w)| &\le C\|\nabla u\|^{1/2}\|u\|^{1/2}\|\nabla v\|\|\nabla w\|, \\ |b(u, v, w)| &\le C\|\nabla u\|\|\nabla v\|\|\nabla w\|^{1/2}\|w\|^{1/2}. \end{aligned} \tag{2.1}$$

The variational formulation for the continuous problem (1.3) is: find $(z^+, z^-, p) : [0, T] \to X \times X \times Q$ satisfying

$$\begin{aligned} &\left(\frac{\partial z^\pm}{\partial t}, v\right) \mp b(B_0, z^\pm, v) + b(z^\mp, z^\pm, v) \\ &\qquad + \nu^+ a(z^\pm, v) + \nu^- a(z^\mp, v) - (p, \bigtriangledown \cdot v) = (f, v) \quad \forall v \in X, \\ &(\bigtriangledown \cdot z^\pm, q) = 0 \quad \forall q \in Q, \end{aligned} \tag{2.2}$$

where $\nu^\pm = (\nu \pm \nu_m)/2$.

Denote X_h and Q_h the finite element spaces for X and Q respectively, built on a conforming, edge to edge triangulation with the maximum triangle parameter denoted by a subscript "h". Likewise, the discrete divergence-free space is denoted by

$$V_h = X_h \cap \{v_h \in X_h : (\bigtriangledown \cdot v_h, q_h) = 0,\ \forall q_h \in Q_h\}.$$

We assume that the finite element spaces satisfy the inverse inequality: $\forall v_h \in X_h$,

$$h\|\nabla v_h\| \le C_{INV}\|v_h\|.$$

The semi-discrete approximation of (2.2) is to find (z_h^+, z_h^-, p_h) : $[0, T] \to X_h \times X_h \times Q_h$ satisfying

$$\begin{aligned} &\left(\frac{\partial z_h^\pm}{\partial t}, v_h\right) \mp b(B_0, z_h^\pm, v_h) + b(z_h^\mp, z_h^\pm, v_h) \\ &\qquad + \nu^+ a(z_h^\pm, v_h) + \nu^- a(z_h^\mp, v_h) - (p_h, \bigtriangledown \cdot v_h) = (f, v_h), \quad \forall v_h \in X_h \\ &(\bigtriangledown \cdot z_h^\pm, q_h) = 0 \quad \forall q_h \in Q_h. \end{aligned} \tag{2.3}$$

3. The partitioned method. The method we propose and analyze herein is a combination of a two-step implicit method with the coupling terms treated by an explicit discretization. Due to the symmetry of the Elsässer fields, we shall use the same time step Δt in both subproblems. The method is described as follows: find $(z^+_{h,n+1}, z^-_{h,n+1}, p^\pm_{h,n+1}) \in X_h \times X_h \times Q_h$, $n \geq 2$, such that $\forall v_h \in X_h$ and $\forall q_h \in Q_h$

$$\begin{aligned}
&\left(\frac{3z^\pm_{h,n+1} - 4z^\pm_{h,n} + z^\pm_{h,n-1}}{2\Delta t}, v_h\right) \mp b(B_0, z^\pm_{h,n+1}, v_h) + b\left((2z^\mp_{h,n} - z^\mp_{h,n-1}), z^\pm_{h,n+1}, v_h\right) \\
&+ \nu^+ a(z^\pm_{h,n+1}, v_h) + \nu^- a((2z^\mp_{h,n} - z^\mp_{h,n-1}, v_h) - (p^\pm_{h,n+1}, \triangledown \cdot v_h) = (f_{n+1}, v_h), \\
&(\triangledown \cdot z^\pm_{h,n+1}, q_h) = 0.
\end{aligned} \tag{3.1}$$

Note that the momentum equations in $z^+_{h,n+1}$ and $z^-_{h,n+1}$ are decoupled, however, the corresponding momentum equations of fluid velocity field u and magnetic field B are not: $\forall v_h \in X_h, q_h \in Q_h$

$$\begin{aligned}
&\left(\frac{3u_{h,n+1} - 4u_{h,n} + u_{h,n-1}}{2\Delta t}, v_h\right) + b(2u_{h,n} - u_{h,n-1}, u_{h,n+1}, v_h) - b(2B_{h,n} - B_{h,n-1}, B_{h,n+1}, v_h) \\
&+ \nu^+ a(u_{h,n+1}, v_h) + \nu^- a(2u_{h,n} - u_{h,n-1}, v_h) - \left(\frac{p^+_{h,n+1} + p^-_{h,n+1}}{2}, \nabla \cdot v_h\right) = (f_{n+1}, v_h), \\
&\left(\frac{3B_{h,n+1} - 4B_{h,n} + B_{h,n-1}}{2\Delta t}, v_h\right) + b(2u_{h,n} - u_{h,n-1}, B_{h,n+1}, v_h) - b(2B_{h,n} - B_{h,n-1}, u_{h,n+1}, v_h) \\
&+ \nu^+ a(B_{h,n+1}, v_h) - \nu^- a(2b_{h,n} - b_{h,n-1}, v_h) - \left(\frac{p^+_{h,n+1} - p^-_{h,n+1}}{2}, \nabla \cdot v_h\right) = 0, \\
&(\triangledown \cdot u_{h,n+1}, q_h) = 0, \qquad (\triangledown \cdot b_{h,n+1}, q_h) = 0.
\end{aligned}$$

3.1. Long-time stability of the partitioned method. The goal of this section is to demonstrate the stability of the method (3.1). It will be shown that the stability depends on the magnetic Prandtl number. More specifically, the partitioned method is *unconditionally stable* for $1/2 < Pr_m < 2$; otherwise it is conditionally stable under the CFL condition (3.13).

LEMMA 3.1. *If $1/2 < Pr_m < 2$, the solution to the partitioned method (3.1) satisfies the following energy identity for any $N \geq 1$,*

$$\begin{aligned}
&\|\mathbf{w}^+_{h,N}\|^2_G + \|\mathbf{w}^-_{h,N}\|^2_G + \frac{1}{2}\sum_{n=1}^{N-1}\left(\|D_2 z^+_{h,n+1}|\|^2 + \|D_2 z^-_{h,n+1}\|^2\right) \\
&+ 2\Delta t\left(\nu^+ - \frac{3|\nu^-|}{2}\right)\left(\|\nabla z^+_{h,N}\|^2 + \|\nabla z^-_{h,N}\|^2\right) + 2\Delta t\left(\nu^+ - \frac{5|\nu^-|}{2}\right)\left(\|\nabla z^+_{h,N-1}\|^2 + \|\nabla z^-_{h,N-1}\|^2\right) \\
&+ 2\Delta t\left(\nu^+ - 3|\nu^-|\right)\sum_{n=2}^{N-2}\left(\|\nabla z^+_{h,n}\|^2 + \|\nabla z^-_{h,n}\|^2\right) + \sum_{n=1}^{N-1}\mathcal{P}_{n+1} \\
&= \|\mathbf{w}^+_{h,1}\|^2_G + \|\mathbf{w}^-_{h,1}\|^2_G + 3|\nu^-|\Delta t\left(\|\nabla z^+_{h,1}\|^2 + \|\nabla z^-_{h,1}\|^2\right) + |\nu^-|\Delta t\left(\|\nabla z^+_{h,0}\|^2 + \|\nabla z^-_{h,0}\|^2\right) \\
&+ 2\Delta t\sum_{n=1}^{N-1}\left((f_{n+1}, z^+_{h,n+1}) + (f_{n+1}, z^-_{h,n+1})\right),
\end{aligned} \tag{3.2}$$

where $\mathbf{w}^\pm_{h,n} = (z^\pm_{h,n}, z^\pm_{h,n-1})^T$ and each $\mathcal{P}_{n+1}$ is a positive term

$$\mathcal{P}_{n+1} = 2\Delta t|\nu^-|\left(\|z^+_{h,n+1} + sign(\nu^-)z^-_{h,n}\|^2 + \|z^-_{h,n+1} + sign(\nu^-)z^+_{h,n}\|^2\right)$$

$$+\Delta t|\nu^-|\left(\|z^+_{h,n+1}-sign(\nu^-)z^-_{h,n-1}\|^2+\|z^-_{h,n+1}-sign(\nu^-)z^+_{h,n-1}\|^2\right).$$

Proof. *Note that $1/2 < Pr_m < 2$ implies $\nu^+ > 3|\nu^-|$. Set $v_h = z^{\pm}_{h,n+1}$ in (3.1), then the coupling terms vanish due to the skew symmetry of the trilinear form b. Thus we obtain*

$$\frac{1}{2\Delta t}\left(\|\mathbf{w}^{\pm}_{h,n+1}\|^2_G-\|\mathbf{w}^{\pm}_{h,n}\|^2_G\right)+\frac{1}{4\Delta t}\|D_2z^{\pm}_{h,n+1}\|^2 \tag{3.3}$$
$$+\nu^+\|\bigtriangledown z^{\pm}_{h,n+1}\|^2+\nu^-a\left((2z^{\mp}_{h,n}-z^{\mp}_{h,n-1}),z^{\pm}_{h,n+1}\right)=\left(f_{n+1},z^{\pm}_{h,n+1}\right).$$

It follows from simple calculation that

$$\nu^-a\left((2z^{\mp}_{h,n}-z^{\mp}_{h,n-1}),z^{\pm}_{h,n+1}\right) \tag{3.4}$$
$$=-\nu^-sign(\nu^-)\left(\|\nabla z^{\pm}_{h,n+1}\|^2+\|\nabla z^{\mp}_{h,n}\|^2-\|\nabla z^{\pm}_{h,n+1}+sign(\nu^-)\nabla z^{\mp}_{h,n}\|^2\right)$$
$$-\frac{\nu^-}{2}sign(\nu^-)\left(\|\nabla z^{\pm}_{h,n+1}\|^2+\|\nabla z^{\mp}_{h,n-1}\|^2-\|\nabla z^{\pm}_{h,n+1}-sign(\nu^-)\nabla z^{\mp}_{h,n-1}\|^2\right)$$
$$=-\frac{3|\nu^-|}{2}\|\nabla z^{\pm}_{h,n+1}\|^2-|\nu^-|\|\nabla z^{\mp}_{h,n}\|^2-\frac{|\nu^-|}{2}\|\nabla z^{\mp}_{h,n-1}\|^2$$
$$+|\nu^-|\|\nabla z^{\pm}_{h,n+1}+sign(\nu^-)\nabla z^{\mp}_{h,n}\|^2+\frac{|\nu^-|}{2}\|\nabla z^{\pm}_{h,n+1}-sign(\nu^-)\nabla z^{\mp}_{h,n-1}\|^2.$$

Combining (3.3) and (3.4) yields

$$\frac{1}{2\Delta t}\left(\|\mathbf{w}^{\pm}_{h,n+1}\|^2_G-\|\mathbf{w}^{\pm}_{h,n}\|^2_G\right)+\frac{1}{4\Delta t}\|D_2z^{\pm}_{h,n+1}\|^2 \tag{3.5}$$
$$+\left(\nu^+-\frac{3|\nu^-|}{2}\right)\|\bigtriangledown z^{\pm}_{h,n+1}\|^2-|\nu^-|\|\nabla z^{\mp}_{h,n}\|^2-\frac{|\nu^-|}{2}\|\nabla z^{\mp}_{h,n-1}\|^2$$
$$+|\nu^-|\|\nabla z^{\pm}_{h,n+1}+sign(\nu^-)\nabla z^{\mp}_{h,n}\|^2+\frac{|\nu^-|}{2}\|\nabla z^{\pm}_{h,n+1}-sign(\nu^-)\nabla z^{\mp}_{h,n-1}\|^2.$$
$$=\left(f_{n+1},z^{\pm}_{h,n+1}\right).$$

By adding the Elsässer fields together and summing up (3.5) from $n=1$ to $N-1$, and finally multiplying by $2\Delta t$, we obtain (3.2) □

THEOREM 3.2. *Assume that $1/2 < Pr_m < 2$, then we have the following results.*
1°. *The partitioned method (3.1) is unconditionally stable, i.e., for any $N \geq 1$ there holds*

$$\|\mathbf{w}^+_{h,N}\|^2_G+\|\mathbf{w}^-_{h,N}\|^2_G+\frac{1}{2}\sum_{n=1}^{N-1}\left(\|D_2z^+_{h,n+1}|\|^2+\|D_2z^-_{h,n+1}\|^2\right) \tag{3.6}$$
$$+\Delta t\left(\nu^+-3|\nu^-|\right)\sum_{n=2}^{N}\left(\|\bigtriangledown z^+_{h,n}\|^2+\|\bigtriangledown z^-_{h,n}\|^2\right)$$
$$\leq\|\mathbf{w}^+_{h,1}\|^2_G+\|\mathbf{w}^-_{h,1}\|^2_G+\frac{\Delta tC_p^2}{(\nu^+-3|\nu^-|)}\sum_{n=1}^{N-1}\|f_{n+1}\|^2$$
$$+3\Delta t|\nu^-|\left(\|\bigtriangledown z^+_{h,1}\|^2+\|\bigtriangledown z^-_{h,1}\|^2+\|\bigtriangledown z^+_{h,0}\|^2+\|\bigtriangledown z^-_{h,0}\|^2\right).$$

2°. *Assuming $f \in L^\infty(0,T;L^2(\Omega))$, the solution is uniformly bounded for all time: there exist $0<\lambda_1<1$, $0<\lambda_2<\infty$ such that*

$$\|z^+_{h,N}\|^2+\|z^-_{h,N}\|^2\leq\lambda_1^NE_1+\lambda_2, \tag{3.7}$$

where

$$E_1 = \|\mathbf{w}^+_{h,1}\|^2_G + \|\mathbf{w}^-_{h,1}\|^2_G + \frac{\Delta t(\nu^+ + 3|\nu^-|)}{2}\left(\| \bigtriangledown z^+_{h,1}\|^2 + \| \bigtriangledown z^-_{h,1}\|^2\right)$$
$$+ \Delta t|\nu^-| \left(+\| \bigtriangledown z^+_{h,0}\|^2 + \| \bigtriangledown z^-_{h,0}\|^2\right).$$

3°. *In addition, if* $f \equiv 0$*, then*

$$z^+_{h,N} \to 0 \quad \textit{and} \quad z^-_{h,N} \to 0 \tag{3.8}$$

in $H^1(\Omega)$ *as* $N \to \infty$*.*

Proof. 1°. Thanks to the Poincaré inequality and Young's inequality, the forcing terms in (3.2) are bounded by

$$2\Delta t\left(f_{n+1}, z^\pm_{h,n+1}\right) \le \Delta t(\nu^+ - 3|\nu^-|)\|\nabla z^\pm_{h,n+1}\|^2 + \frac{\Delta t C_p^2}{(\nu^+ - 3|\nu^-|)}\|f_{n+1}\|^2. \tag{3.9}$$

Then (3.6) follows from (3.9) and by dropping the positive term $\mathcal{P}_N$.

2°. Combining (3.5) and (3.9) gives

$$\frac{1}{2}\left(\|\mathbf{w}^+_{h,n+1}\|^2_G + \|\mathbf{w}^-_{h,n+1}\|^2_G\right) + \Delta t\nu^+\left(\| \bigtriangledown z^+_{h,n+1}\|^2 + \| \bigtriangledown z^-_{h,n+1}\|^2\right)$$
$$\le \frac{1}{2}\left(\|\mathbf{w}^+_{h,n}\|^2_G + \|\mathbf{w}^-_{h,n}\|^2_G\right) + 2\Delta t|\nu^-|\left(\| \bigtriangledown z^+_{h,n}\|^2 + \| \bigtriangledown z^-_{h,n}\|^2\right)$$
$$+ \Delta t|\nu^-|\left(\| \bigtriangledown z^+_{h,n-1}\|^2 + \| \bigtriangledown z^-_{h,n-1}\|^2\right) + \frac{\Delta t C_p^2}{(\nu^+ - 3|\nu^-|)}\|f_{n+1}\|^2.$$

Add $\frac{\Delta t(\nu^+ - |\nu^-|)}{2}\left(\|\nabla z^+_{h,n}\|^2 + \|\nabla z^-_{h,n}\|^2\right)$ to both sides of the above inequality, we then obtain

$$\frac{1}{2}\left(\|\mathbf{w}^+_{h,n+1}\|^2_G + \|\mathbf{w}^-_{h,n+1}\|^2_G\right) + \Delta t\nu^+\left(\| \bigtriangledown z^+_{h,n+1}\|^2 + \| \bigtriangledown z^-_{h,n+1}\|^2\right) + \frac{\Delta t(\nu^+ - |\nu^-|)}{2}\left(\|\nabla z^+_{h,n}\|^2 + \|\nabla z^-_{h,n}\|^2\right)$$
$$\le \frac{1}{2}\left(\|\mathbf{w}^+_{h,n}\|^2_G + \|\mathbf{w}^-_{h,n}\|^2_G\right) + \frac{\Delta t(\nu^+ + 3|\nu^-|)}{2}|\left(\| \bigtriangledown z^+_{h,n}\|^2 + \| \bigtriangledown z^-_{h,n}\|^2\right)$$
$$+ \Delta t|\nu^-|\left(\| \bigtriangledown z^+_{h,n-1}\|^2 + \| \bigtriangledown z^-_{h,n-1}\|^2\right) + \frac{\Delta t C_p^2}{(\nu^+ - 3|\nu^-|)}\|f_{n+1}\|^2,$$

∎

which is equivalent to

$$E_{n+1} + \frac{\Delta t(\nu^+ - 3|\nu^-|)}{2}\left(\|\nabla z^+_{h,n+1}\|^2 + \|\nabla z^-_{h,n+1}\|^2 + \|\nabla z^+_{h,n}\|^2 + \|\nabla z^-_{h,n}\|^2\right)$$
$$\le E_n + \frac{\Delta t C_p^2}{(\nu^+ - 3|\nu^-|)}\|f_{n+1}\|^2, \tag{3.10}$$

where

$$E_n = \frac{1}{2}\left(\|\mathbf{w}^+_{h,n}\|^2_G + \|\mathbf{w}^-_{h,n}\|^2_G\right) + \frac{\Delta t(\nu^+ + 3|\nu^-|)}{2}\left(\| \bigtriangledown z^+_{h,n}\|^2 + \| \bigtriangledown z^-_{h,n}\|^2\right)$$
$$+ \Delta t|\nu^-|\left(\| \bigtriangledown z^+_{h,n-1}\|^2 + \| \bigtriangledown z^-_{h,n-1}\|^2\right).$$

Applying the Poincaré inequality and the equivalence of G-norm and L^2-norm, we have

$$
\begin{aligned}
&\frac{\Delta t(\nu^+ - 3|\nu^-|)}{2}\left(\|\nabla z^+_{h,n+1}\|^2 + \|\nabla z^-_{h,n+1}\|^2 + \|\nabla z^+_{h,n}\|^2 + \|\nabla z^-_{h,n}\|^2\right)\\
&\geq \frac{\Delta t(\nu^+ - 3|\nu^-|)}{4}\left(\|\nabla z^+_{h,n+1}\|^2 + \|\nabla z^-_{h,n+1}\|^2 + \|\nabla z^+_{h,n}\|^2 + \|\nabla z^-_{h,n}\|^2\right)\\
&+ \frac{\Delta t(\nu^+ - 3|\nu^-|)}{4\beta^2 C_p^2}\left(\|\mathbf{w}^+_{h,n+1}\|^2 + \|\mathbf{w}^-_{h,n+1}\|^2\right).
\end{aligned}
$$

Thus, by setting $C_1 = \min\left\{\frac{\nu^+ - 3|\nu^-|}{2\Delta t(\nu^+ + 3|\nu^-|)}, \frac{(\nu^+ - 3|\nu^-|)}{2(3-2\sqrt{2})^2 C_p^2}\right\}$, we have from (3.10) that

$$
(1 + C_1\Delta t)E_{n+1} \leq E_n + \frac{\Delta t C_p^2}{(\nu^+ - 3|\nu^-|)}\|f_{n+1}\|^2,
$$

which, by induction, implies

$$
E_{n+1} \leq \frac{E_1}{(1 + C_1\Delta t)^n} + \frac{C_p^2(1 + C_1\Delta t)}{C_1(\nu^+ - 3|\nu^-|)}\max_i \|f_i\|^2.
$$

Setting

$$
\lambda_1 = \frac{1}{1 + C_1\Delta t}, \quad \lambda_2 = \frac{C_p^2(1 + C_1\Delta t)}{C_1(\nu^+ - 3|\nu^-|)}\max_i \|f_i\|^2,
$$

we complete the second part (3.7).

3°. Finally, if $f \equiv 0$, the series in (3.6)

$$
\sum_{n=2}^{\infty}\left(\| \bigtriangledown z^+_{h,n}\|^2 + \| \bigtriangledown z^-_{h,n}\|^2\right)
$$

converges and therefore (3.8) follows. □

Lemma 3.3. *If $Pr \geq 2$ or $Pr \leq 1/2$, the solution to the partitioned method (3.1) satisfies the following energy identity for any $N \geq 1$,*

$$
\begin{aligned}
&\|\mathbf{w}^+_{h,N}\|_G^2 + \|\mathbf{w}^-_{h,N}\|_G^2 + \frac{3\Delta t}{2}(\nu^+ - |\nu^-|)\sum_{n=1}^{N-1}\left(\|\nabla z^+_{h,n+1}\|^2 + \|\nabla z^-_{h,n+1}\|^2\right) + \sum_{n=1}^{N-1}\tilde{\mathcal{P}}_{n+1} \qquad (3.11)\\
&+ \frac{1}{2}\sum_{n=1}^{N-1}\left(\|D_2 z^+_{h,n+1}\||^2 + \|D_2 z^-_{h,n+1}\|^2\right) - \frac{2\Delta t|\nu^-|^2}{(\nu^+ - |\nu^-|)}\sum_{n=1}^{N-1}\left(\| \bigtriangledown D_2 z^+_{h,n+1}\|^2 + \| \bigtriangledown D_2 z^-_{h,n+1}\|^2\right)\\
&= \|\mathbf{w}^+_{h,1}\|_G^2 + \|\mathbf{w}^-_{h,1}\|_G^2 + 2\Delta t\sum_{n=1}^{N-1}\left((f_{n+1}, z^+_{h,n+1}) + (f_{n+1}, z^-_{h,n+1})\right),
\end{aligned}
$$

where each $\tilde{\mathcal{P}}_{n+1}$ is a positive term

$$
\begin{aligned}
\tilde{\mathcal{P}}_{n+1} &= 2\Delta t|\nu^-|\left(\|\nabla z^-_{h,n+1} + sign(\nu^-) \bigtriangledown z^+_{h,n+1}\|^2 + \|\nabla z^+_{h,n+1} + sign(\nu^-) \bigtriangledown z^-_{h,n+1}\|^2\right)\\
&+ 2\Delta t\left\|\frac{\sqrt{\nu^+ - \nu^-}}{2}\nabla z^+_{h,n+1} - sign(\nu^-)\frac{|\nu^-|}{\sqrt{\nu^+ - \nu^-}}\nabla D_2 z^-_{h,n+1}\right\|\\
&+ 2\Delta t\left\|\frac{\sqrt{\nu^+ - \nu^-}}{2}\nabla z^-_{h,n+1} - sign(\nu^-)\frac{|\nu^-|}{\sqrt{\nu^+ - \nu^-}}\nabla D_2 z^+_{h,n+1}\right\|.
\end{aligned}
$$

Proof. If $Pr \geq 2$ or $Pr \leq 1/2$ there no longer holds $\nu^+ > 3\nu^-$. Thus in this case, the term associated with ν^- in (3.3) is equivalent to

$$
\begin{aligned}
&\nu^- a\left((2z^{\mp}_{h,n} - z^{\mp}_{h,n-1}), z^{\pm}_{h,n+1}\right)\\
&= \nu^-\left(\triangledown z^{\mp}_{h,n+1}, \triangledown z^{\pm}_{h,n+1}\right) - \nu^-\left(\triangledown D_2 z^{\mp}_{h,n+1}, \triangledown z^{\pm}_{h,n+1}\right)\\
&= -\frac{|\nu^-|}{2}\left(\| \triangledown z^{\mp}_{h,n+1}\|^2 + \| \triangledown z^{\pm}_{h,n+1}\|^2\right) + \frac{|\nu^-|}{2}\|\nabla z^{\mp}_{h,n+1} + sign(\nu^-) \triangledown z^{\pm}_{h,n+1}\|^2\\
&- \frac{\nu^+ - |\nu^-|}{4}\| \triangledown z^{\pm}_{h,n+1}\|^2 - \frac{|\nu^-|^2}{\nu^+ - |\nu^-|}\| \triangledown D_2 z^{\mp}_{h,n+1}\|^2 + \left\| \frac{\sqrt{\nu^+ - \nu^-}}{2}\nabla z^{\pm}_{h,n+1} - sign(\nu^-)\frac{|\nu^-|}{\sqrt{\nu^+ - \nu^-}}\nabla D_2 z^{\mp}_{h,n+1}\right\|.
\end{aligned}
$$

Applying the above equality to (3.3) gives

$$
\begin{aligned}
&\|\mathbf{w}^+_{h,n+1}\|^2_G + \|\mathbf{w}^-_{h,n+1}\|^2_G + \frac{3\Delta t}{2}(\nu^+ - |\nu^-|)\left(\|\nabla z^+_{h,n+1}\|^2 + \|\nabla z^-_{h,n+1}\|^2\right) + \tilde{\mathcal{P}}_{n+1}\\
&+ \frac{1}{2}\left(\|D_2 z^+_{h,n+1}|\|^2 + \|D_2 z^-_{h,n+1}\|^2\right) - \frac{2\Delta t|\nu^-|^2}{(\nu^+ - |\nu^-|)}\left(\| \triangledown D_2 z^+_{h,n+1}\|^2 + \| \triangledown D_2 z^-_{h,n+1}\|^2\right)\\
&= \|\mathbf{w}^+_{h,n}\|^2_G + \|\mathbf{w}^-_{h,n}\|^2_G + 2\Delta t\left((f_{n+1}, z^+_{h,n+1}) + (f_{n+1}, z^-_{h,n+1})\right).
\end{aligned}
\tag{3.12}
$$

Therefore, (3.11) follows by summing (3.12) from $n = 1$ to $N - 1$. □

THEOREM 3.4. *If $Pr \geq 2$ or $Pr \leq 1/2$ the partitioned method (3.1) is stable under the CFL condition*

$$
\Delta t \leq \frac{\nu^+ - |\nu^-|}{4C^2_{INV}|\nu^-|^2}h^2. \tag{3.13}
$$

More precisely, for any $N \geq 1$ there holds

$$
\begin{aligned}
&\|\mathbf{w}^+_{h,N}\|^2_G + \|\mathbf{w}^-_{h,N}\|^2_G + \Delta t(\nu^+ - |\nu^-|)\sum_{n=1}^{N-1}\left(\|\nabla z^+_{h,n+1}\|^2 + \|\nabla z^-_{h,n+1}\|^2\right)\\
&\leq \|\mathbf{w}^+_{h,1}\|^2_G + \|\mathbf{w}^-_{h,1}\|^2_G + \frac{2\Delta t C^2_p}{\nu^+ - |\nu^-|}\sum_{n=1}^{N-1}\|f_{n+1}\|^2.
\end{aligned}
\tag{3.14}
$$

If $f \in L^\infty(0,T;L^2(\Omega))$, then the solution is uniformly bounded for all time. In addition, if $f \equiv 0$, then

$$
z^+_{h,N} \to 0 \quad \textit{and} \quad z^-_{h,N} \to 0 \tag{3.15}
$$

in $H^1(\Omega)$ as $N \to \infty$.

Proof. Thanks to the inverse inequality, we have

$$
\begin{aligned}
&\frac{1}{2}\|D_2 z^{\pm}_{h,n+1}|\|^2 - \frac{2\Delta t|\nu^-|^2}{(\nu^+ - |\nu^-|)}\| \triangledown D_2 z^+_{h,n+1}\|^2\\
&\geq \frac{h^2}{2C^2_{INV}}\|\nabla D_2 z^{\pm}_{h,n+1}|\|^2 - \frac{2\Delta t|\nu^-|^2}{(\nu^+ - |\nu^-|)}\| \triangledown D_2 z^+_{h,n+1}\|^2\\
&= \left(\frac{h^2}{2C^2_{INV}} - \frac{2\Delta t|\nu^-|^2}{(\nu^+ - |\nu^-|)}\right)\| \triangledown D_2 z^+_{h,n+1}\|^2.
\end{aligned}
\tag{3.16}
$$

The forcing terms are bounded due to the Poincaré inequality and Young's inequality

$$
2\Delta t\left(f_{n+1}, z^{\pm}_{h,n+1}\right) \leq \frac{\Delta t(\nu^+ - |\nu^-|)}{2}\|\nabla z^{\pm}_{h,n+1}\|^2 + \frac{2\Delta t C^2_p}{\nu^+ - |\nu^-|}\|f_{n+1}\|^2. \tag{3.17}
$$

Combining (3.12), (3.16), (3.17) and dropping $\tilde{\mathcal{P}}_{n+1}$ yields

$$\begin{aligned}
&\|\mathbf{w}^{+}_{h,n+1}\|^2_G + \|\mathbf{w}^{-}_{h,n+1}\|^2_G + \Delta t(\nu^+ - |\nu^-|)\left(\|\nabla z^{+}_{h,n+1}\|^2 + \|\nabla z^{-}_{h,n+1}\|^2\right) \\
&+ \left(\frac{h^2}{2C^2_{INV}} - \frac{2\Delta t|\nu^-|^2}{(\nu^+ - |\nu^-|)}\right)\| \nabla D_2 z^{+}_{h,n+1}\|^2 \\
&\le \|\mathbf{w}^{+}_{h,n}\|^2_G + \|\mathbf{w}^{-}_{h,n}\|^2_G + \frac{2\Delta t C_p^2}{\nu^+ - |\nu^-|}\|f_{n+1}\|^2,
\end{aligned} \tag{3.18}$$

In which the numerical dissipation term is positive under the CFL condition (3.13). By a telescope sum, we obtain (3.14).

Next, we add $\frac{\Delta t(\nu^+ - |\nu^-|)}{2}\left(\|\nabla z^{+}_{h,n}\|^2 + \|\nabla z^{-}_{h,n}\|^2\right)$ to both sides of (3.18) and let

$$\begin{aligned}
E_n = \frac{1}{2}\left(\|\mathbf{w}^{+}_{h,n}\|^2_G + \|\mathbf{w}^{-}_{h,n}\|^2_G\right) &+ \frac{\Delta t(\nu^+ - |\nu^-|)}{2}\left(\| \nabla z^{+}_{h,n}\|^2 + \| \nabla z^{-}_{h,n}\|^2\right) \\
&+ \frac{\Delta t(\nu^+ - |\nu^-|)}{4}\left(\| \nabla z^{+}_{h,n-1}\|^2 + \| \nabla z^{-}_{h,n-1}\|^2\right).
\end{aligned}$$

Then (3.12) implies

$$\begin{aligned}
&E_{n+1} + \frac{\Delta t(\nu^+ - |\nu^-|)}{2}\left(\| \nabla z^{+}_{h,n}\|^2 + \| \nabla z^{-}_{h,n}\|^2\right) + \frac{\Delta t(\nu^+ - |\nu^-|)}{4}\left(\| \nabla z^{+}_{h,n-1}\|^2 + \| \nabla z^{-}_{h,n-1}\|^2\right) \\
&\le E_n + \frac{2\Delta t C_p^2}{\nu^+ - |\nu^-|}\|f_{n+1}\|^2.
\end{aligned}$$

The rest of the proof follows analogously as in Theorem 3.2, i.e., utilizing the Poincaré inequality and the equivalence of G-norm and L^2-norm. ☐

REMARK 3.1. *The time step condition (3.13) is reasonably large for Pr_m is of order unity. To this end, we write the time step condition as follows,*

$$\frac{\Delta t}{h^2} \le \frac{\min\{\nu, \nu_m\}}{C^2_{INV}(Pr_m - 1)^2\nu_m^2},$$

which implies that

$$\frac{\Delta t}{h^2} \sim \mathcal{O}\left(\frac{1}{\nu_m}\right), \quad \text{for } Pr_m \sim \mathcal{O}(1).$$

4. Error analysis. In this section, we study the convergence of the method (3.1), where spatial discretization is effected using finite element methods. Recall that our finite element spaces satisfy the discrete inf-sup conditions. To establish the optimal error estimates for the approximation, we assume that the true solutions satisfy regularity conditions

$$\begin{aligned}
&z^{\pm} \in L^{\infty}\left(0, T; (H^{k+1}(\Omega))^d\right) \cap H^1\left(0, T; (H^{k+1}(\Omega))^d\right) \cap H^2\left(0, T; (H^{k+1}(\Omega))^d\right), \\
&p \in L^2\left(0, T; (H^{s+1}(\Omega))^d\right).
\end{aligned} \tag{4.1}$$

The errors are denoted by $e^{\pm}_n = z^{\pm}_n - z^{\pm}_{h,n}$. Similar to the stability analysis, the error estimate depends on the values of magnetic Prandtl number in the sense of time step restriction. Nevertheless, the convergence rate is the same with respect to the mesh size and time step in both situations. For the sake of readability, the proofs are given in the Appendix.

THEOREM 4.1. *Assume that the Prandtl number $1/2 < Pr_m < 2$, and suppose that $(z^{\pm}, p)$ satisfies the weak formulation (2.2) and regularity conditions (4.1). If $(z_h^{\pm}, p_h^{\pm})$ is given by the algorithm (3.1) with $n \in \{1, 2, \cdots, T/\Delta t\}$, we have the following error estimate.*

$$
\begin{aligned}
&\frac{1}{2}\left(\|e_n^+\|^2 + \|e_n^-\|^2 + \|2e_n^+ - e_{n-1}^+\|^2 + \|2e_n^- - e_{n-1}^-\|^2\right) \qquad (4.2)\\
&+ \Delta t(\nu^+ - 3|\nu^-|)\sum_{j=2}^{n}\left(\|\bigtriangledown e_j^+\|^2 + \|\bigtriangledown e_j^-\|^2\right) + \frac{1}{2}\sum_{j=1}^{n-1}\left(\|D_2 e_{j+1}^+\|^2 + \|D_2 e_{j+1}^-\|^2\right)\\
&\leq C_0\Bigg\{\|z_1^+ - z_{h,1}^+\|^2 + \|z_1^- - z_{h,1}^-\|^2 + \|z_0^+ - z_{h,0}^+\|^2 + \|z_0^- - z_{h,0}^-\|^2\\
&+ \|\bigtriangledown(z_1^+ - z_{h,1}^+)\|^2 + \|\bigtriangledown(z_1^- - z_{h,1}^-)\|^2 + \|\bigtriangledown(z_0^+ - z_{h,0}^+)\|^2 + \|\bigtriangledown(z_0^- - z_{h,0}^-)\|^2\\
&+ h^{2k+2}|||z^+|||_{\infty,k+1}^2 + h^{2k+2}|||z^-|||_{\infty,k+1}^2\\
&+ \Delta t^4\|z_{ttt}^+\|_{2,0}^2 + +\Delta t^4\|z_{ttt}^-\|_{2,0}^2 + \Delta t^4\|\bigtriangledown z_{tt}^+\|_{2,0}^2 + \Delta t^4\|\bigtriangledown z_{tt}^-\|_{2,0}^2 + h^{2s+2}|||p|||_{2,s+1}^2\\
&+ h^{2k}|||z^+|||_{2,k+1}^2 + h^{2k}|||z^-|||_{2,k+1}^2 + h^{2k}|||z^+|||_{4,k+1}^4 + h^{2k}|||z^-|||_{4,k+1}^4\\
&+ h^{2k+2}\|z_t^+\|_{2,k+1}^2\} + h^{2k+2}\|z_t^-\|_{2,k+1}^2\Bigg\}.
\end{aligned}
$$

THEOREM 4.2. *Assume that the Prandtl number $Pr_m \leq 1/2$ or $Pr_m \geq 2$. Then under the CFL condition (3.13), we have the error estimates*

$$
\begin{aligned}
&\frac{1}{2}\left(\|e_n^+\|^2 + \|e_n^-\|^2 + \|2e_n^+ - e_{n-1}^+\|^2 + \|2e_n^- - e_{n-1}^-\|^2\right) \qquad (4.3)\\
&+ \Delta t(\nu^+ - |\nu^-|)\sum_{j=2}^{n}\left(\|\bigtriangledown e_j^+\|^2 + \|\bigtriangledown e_j^-\|^2\right) + \frac{1}{4}\sum_{j=1}^{n-1}\left(\|D_2 e_{j+1}^+\|^2 + \|D_2 e_{j+1}^-\|^2\right)\\
&\leq C_0\Bigg\{\|z_1^+ - z_{h,1}^+\|^2 + \|z_1^- - z_{h,1}^-\|^2 + \|z_0^+ - z_{h,0}^+\|^2 + \|z_0^- - z_{h,0}^-\|^2\\
&+ \|\bigtriangledown(z_1^+ - z_{h,1}^+)\|^2 + \|\bigtriangledown(z_1^- - z_{h,1}^-)\|^2 + \|\bigtriangledown(z_0^+ - z_{h,0}^+)\|^2 + \|\bigtriangledown(z_0^- - z_{h,0}^-)\|^2\\
&+ h^{2k+2}|||z^+|||_{\infty,k+1}^2 + h^{2k+2}|||z^-|||_{\infty,k+1}^2\\
&+ \Delta t^4\|z_{ttt}^+\|_{2,0}^2 + +\Delta t^4\|z_{ttt}^-\|_{2,0}^2 + \Delta t^4\|\bigtriangledown z_{tt}^+\|_{2,0}^2 + \Delta t^4\|\bigtriangledown z_{tt}^-\|_{2,0}^2 + h^{2s+2}|||p|||_{2,s+1}^2\\
&+ h^{2k}|||z^+|||_{2,k+1}^2 + h^{2k}|||z^-|||_{2,k+1}^2 + h^{2k}|||z^+|||_{4,k+1}^4 + h^{2k}|||z^-|||_{4,k+1}^4\\
&+ h^{2k+2}\|z_t^+\|_{2,k+1}^2\} + h^{2k+2}\|z_t^-\|_{2,k+1}^2\Bigg\}.
\end{aligned}
$$

Consequently, for Taylor-Hood elements, i.e., $k = 2,\ s = 1$, we have the following result.

COROLLARY 4.3. *Suppose that (X^h, Q^h) is given by P_2-P_1 Taylor-Hood approximation elements, i.e., piecewise quadratic finite elements for $z_h^{\pm}$ and piecewise linear finite elements for $p_h^{\pm}$. Then there is a positive constant C_0 such that*

$$
|||e^+|||_{\infty,0}^2 + |||e^-|||_{\infty,0}^2 + |||\bigtriangledown e^+|||_{2,0}^2 + |||\bigtriangledown e^-|||_{2,0}^2 \leq C_0(\Delta t^4 + h^4). \qquad (4.4)
$$

5. Numerical tests. In this section we verify the rate of convergence of the method (3.1) on an electrically conducted two-dimensional traveling wave problem [34]. The true solutions (in Elsässer variables) are

$$z^+ = \begin{pmatrix} \frac{3}{4} + \frac{1}{4}\cos(2\pi(x-t))\sin(2\pi(y-t))e^{-8\pi^2\nu t} + \frac{1}{10}(y+1)^2 e^{\nu_m t} \\ -\frac{1}{4}\sin(2\pi(x-t))\cos(2\pi(y-t))e^{-8\pi^2\nu t} + \frac{1}{10}(x+1)^2 e^{\nu_m t} \end{pmatrix},$$

$$z^- = \begin{pmatrix} \frac{3}{4} + \frac{1}{4}\cos(2\pi(x-t))\sin(2\pi(y-t))e^{-8\pi^2\nu t} - \frac{1}{10}(y+1)^2 e^{\nu_m t} \\ -\frac{1}{4}\sin(2\pi(x-t))\cos(2\pi(y-t))e^{-8\pi^2\nu t} - \frac{1}{10}(x+1)^2 e^{\nu_m t} \end{pmatrix},$$

$$p = -\frac{1}{64}\left(\cos(4\pi(x-t)) + \cos(4\pi(y-t))\right) e^{-16\pi^2\nu t},$$

defined on the domain $\Omega = [0.5, 1.5]^2$. The kinematic viscosity and magnetic resistivity are set to $\nu = \nu_m = 2.5 \times 10^{-4}$ so that $Pr_m = 1$. The time interval is $0 \le t \le 1$. We adopt piecewise quadratic finite elements for $z_h^{\pm}$ and piecewise linear finite elements for $p_h^{\pm}$. The initial data and source terms are chosen to correspond the exact solutions. According to the convergence analysis (4.4), the errors are second order with respect to the mesh size h and time step Δt. Therefore we take $\Delta t = h$ to easily observe the convergence.

Table 5.1 presents and confirms the rate of convergence provided by Corollary 4.3, where $\|\cdot\|_\infty = \|\cdot\|_{L^\infty(0,T;L^2(\Omega))}$ and $\|\cdot\|_2 = \|\cdot\|_{L^2(0,T;L^2(\Omega))}$.

Figure 5.1 shows the log-log plot of the error for BEFE, SDC and the algorithm (3.1). Interestingly, the rate of convergence of the SDC method is slightly less than two (2).

$\Delta t = h$	$\|z^+ - z_h^+\|_\infty$	rate	$\|\nabla z^+ - \nabla z_h^+\|_2$	rate	$\|z^- - z_h^-\|_\infty$	rate	$\|\nabla z^- - \nabla z_h^-\|_2$	rate
1/16	4.047e-2	–	2.978e+0	–	3.653e-2	–	2.028e+0	–
1/32	6.701e-3	2.59	8.755e-1	1.77	8.536e-3	2.10	7.035e-1	1.53
1/64	1.360e-3	2.30	1.676e-1	2.38	2.101e-3	2.02	1.812e-1	1.96
1/128	3.359e-4	2.02	2.930e-2	2.51	5.217e-4	2.01	4.497e-2	2.01

Table 5.1: Convergence rate for algorithm (3.1).

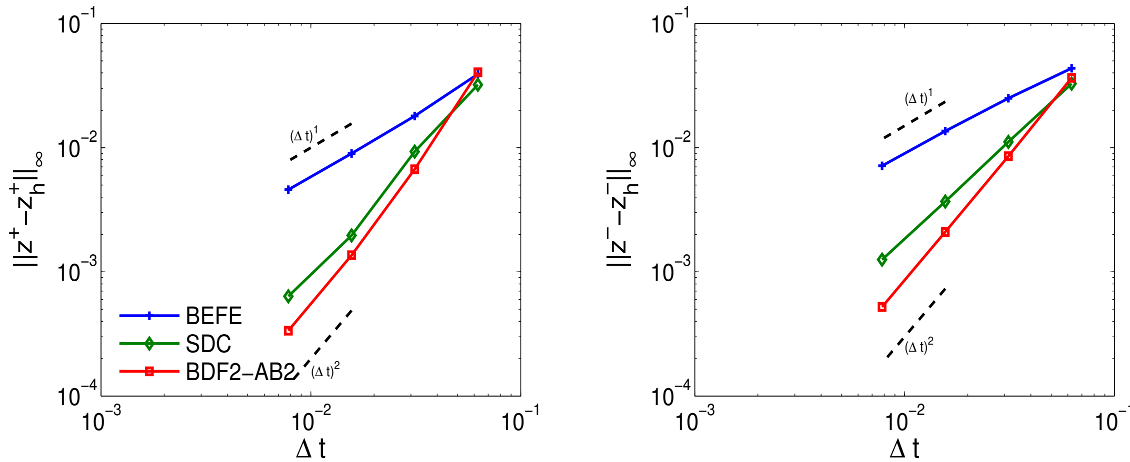

Fig. 5.1: Log-log plot of the error in Elsässer fields as a function of time step Δt.

6. Conclusion. The evolutionary coupled MHD studies the dynamics of the electrically conducting fluids and the electromagnetic fields. When solving a fully coupled MHD, it is usually computationally expensive to use a monolithic method. In contrast, a partitioned method is an attractive approach due to the decoupling of the subproblems and thus, the ability to solve them in parallel. However, to the best of our knowledge, there has been less work dedicated to partitioned methods. Perhaps, this is because of the complex (self- and cross-) couplings of the fluid velocity field and the magnetic field, which may impose a restrictive stability condition on the partitioned methods compared with monolithic methods. The Elsässer fields merges the physical properties of the Navier-Stokes and Maxwell equations. An interesting property of the Elsässer fields is that it contain only the cross-coupling between z^+ and z^-, which may suggest the use of partitioned methods. In fact, it turns out that a partitioned method in Elsässer fields is no longer a partitioned method in the fluid velocity field and magnetic field.

In this paper, we propose a such method (3.1) applied on the Elsässer fields, aiming to reduce the computational complexity. We present a complete analysis on the long-time stability and error estimate. Depending on the magnetic Prandtl number, the algorithm is may or may not unconditionally stable. Nevertheless, the convergence of the error coincides in both situations.

Many open problems remain, such as developing more stable partitioned methods for large or small magnetic Prandtl number, and preserving the divergence-free condition of the magnetic field on the discrete level.

7. Appendix A. *Proof.* [Proof of Theorem 4.1]
The true solution $(z^{\pm}, p)$ at time t_{n+1} satisfies

$$\begin{aligned}&\left(\frac{3z^{\pm}_{n+1} - 4z^{\pm}_{n} + z^{\pm}_{n-1}}{2\Delta t}, v_h\right) \mp b\left(B_0, z^{\pm}_{n+1}, v_h\right) + b\left(z^{\mp}_{n+1}, z^{\pm}_{n+1}, v_h\right)\\ &\quad + \nu^+ a\left(z^{\pm}_{n+1}, v_h\right) + \nu^- a\left(z^{\mp}_{n+1}, v_h\right) - \left(p_{n+1}, \bigtriangledown \cdot v_h\right) = \left(r^{\pm}_{n+1}, v_h\right),\\ &\left(\bigtriangledown \cdot z^{\pm}_{n+1}, q_h\right) = 0,\end{aligned} \tag{7.1}$$

where

$$r^{\pm}_{n+1} := \frac{3z^{\pm}_{n+1} - 4z^{\pm}_{n} + z^{\pm}_{n-1}}{2\Delta t} - \frac{\partial z^{\pm}_{n+1}}{\partial t}.$$

Let $e^{\pm}_{n+1} = z^{\pm}_{n+1} - z^{\pm}_{h,n+1}$ denote the error. We decompose it as

$$e^{\pm}_{n+1} = (z^{\pm}_{n+1} - \tilde{z}^{\pm}_{n+1}) + (\tilde{z}^{\pm}_{n+1} - z^{\pm}_{h,n+1}) =: \eta^{\pm}_{n+1} + \xi^{\pm}_{h,n+1},$$

where $\tilde{z}^{\pm}_{h,n+1}$ is the interpolation of $z^{\pm}_{n+1}$ onto V^h. For notational simplicity, we denote $\xi^{\pm}_{h,n+1} = \xi^{\pm}_{n+1}$. Subtract (3.1) from (7.1) and set $v_h = \xi^{\pm}_{n+1}$, we obtain

$$\begin{aligned}
&\frac{1}{4\Delta t}\left(\|\xi^{\pm}_{n+1}\|^2 + \|2\xi^{\pm}_{n+1} - \xi^{\pm}_{n}\|^2\right) - \frac{1}{4\Delta t}\left(\|\xi^{\pm}_{n}\|^2 + \|2\xi^{\pm}_{n} - \xi^{\pm}_{n-1}\|^2\right) \\
&+ \frac{1}{4\Delta t}\|D_2\xi^{\pm}_{n+1}\|^2 + \nu^+\|\nabla \xi^{\pm}_{n+1}\|^2 \\
&= -\left(\frac{3\eta^{\pm}_{n+1} - 4\eta^{\pm}_{n} + \eta^{\pm}_{n-1}}{2\Delta t}, \xi^{\pm}_{n+1}\right) + \left(r^{\pm}_{n+1}, \xi^{\pm}_{n+1}\right) + \left(p_{n+1} - p^{\pm}_{h,n+1}, \nabla\cdot\xi^{\pm}_{n+1}\right) \\
&+ \mathcal{N}^{\pm}_{n+1} - \mathcal{M}^{\pm}_{n+1}
\end{aligned} \tag{7.2}$$

where

$$\mathcal{N}^{\pm}_{n+1} := \pm b\left(B_0, \eta^{\pm}_{n+1}, \xi^{\pm}_{n+1}\right) - b\left(z^{\mp}_{n+1}, z^{\pm}_{n+1}, \xi^{\pm}_{n+1}\right) + b\left(2z^{\mp}_{h,n} - z^{\mp}_{h,n-1}, z^{\pm}_{n+1}, \xi^{\pm}_{n+1}\right),$$

and

$$\mathcal{M}^{\pm}_{n+1} := \nu^+ a\left(\eta^{\pm}_{n+1}, \xi^{\pm}_{n+1}\right) + \nu^- a\left(z^{\mp}_{n+1}, \xi^{\pm}_{n+1}\right) - \nu^- a\left((2z^{\mp}_{h,n} - z^{\mp}_{h,n-1}), \xi^{\pm}_{n+1}\right).$$

We bound the first three terms on the right-hand side of (7.2) as follows,

$$-\left(\frac{3\eta^{\pm}_{n+1} - 4\eta^{\pm}_{n} + \eta^{\pm}_{n-1}}{2\Delta t}, \xi^{\pm}_{n+1}\right) \le \varepsilon\|\nabla \xi^{\pm}_{n+1}\|^2 + \frac{C^2}{4\varepsilon\Delta t}\int_{t_{n-1}}^{t_{n+1}} \|\eta^{\pm}_{t}\|^2 dt, \tag{7.3}$$

$$(r^{\pm}_{n+1}, \xi^{\pm}_{n+1}) \le \varepsilon\|\nabla \xi^{\pm}_{n+1}\|^2 + \frac{C^2\Delta t^3}{4\varepsilon}\int_{t_{n-1}}^{t_{n+1}} \|z^{\pm}_{ttt}\|^2 dt, \tag{7.4}$$

and

$$\left(p_{n+1} - p^{\pm}_{h,n+1}, \nabla\cdot\xi^{\pm}_{n+1}\right) \le \varepsilon\|\nabla \xi^{\pm}_{n+1}\|^2 + \frac{C^2}{4\varepsilon}\|p_{n+1} - p^{\pm}_{h,n+1}\|^2. \tag{7.5}$$

By adding and subtracting

$$b(2z^{\mp}_{n} - z^{\mp}_{n-1}, z^{\pm}_{n+1}, \xi^{\pm}_{n+1}) \quad \text{and} \quad b(2z^{\mp}_{h,n} - z^{\mp}_{h,n-1}, z^{\pm}_{n+1}, \xi^{\pm}_{n+1})$$

to the nonlinear term $\mathcal{N}^{\pm}_{n+1}$, it follows that

$$\begin{aligned}
\mathcal{N}^{\pm}_{n+1} &= \pm b\left(B_0, \eta^{\pm}_{n+1}, \xi^{\pm}_{n+1}\right) - b\left(D_2 z^{\mp}_{n+1}, z^{\pm}_{n+1}, \xi^{\pm}_{n+1}\right) \\
&\quad - b\left(2e^{\mp}_{n} - e^{\mp}_{n-1}, z^{\pm}_{n+1}, \xi^{\pm}_{n+1}\right) - b\left(2z^{\mp}_{h,n} - z^{\mp}_{h,n-1}, e^{\pm}_{n+1}, \xi^{\pm}_{n+1}\right) \\
&= \pm b\left(B_0, \eta^{\pm}_{n+1}, \xi^{\pm}_{n+1}\right) - b\left(D_2 z^{\mp}_{n+1}, z^{\pm}_{n+1}, \xi^{\pm}_{n+1}\right) \\
&\quad - b\left(2e^{\mp}_{n}, z^{\pm}_{n+1}, \xi^{\pm}_{n+1}\right) + b\left(e^{\mp}_{n-1}, z^{\pm}_{n+1}, \xi^{\pm}_{n+1}\right) \\
&\quad - b\left(2z^{\mp}_{h,n}, e^{\pm}_{n+1}, \xi^{\pm}_{n+1}\right) + b\left(z^{\mp}_{h,n-1}, e^{\pm}_{n+1}, \xi^{\pm}_{n+1}\right).
\end{aligned}$$

Using (2.1), each term above is estimated as

$$\pm b\left(B_0,\eta_{n+1}^{\pm},\xi_{n+1}^{\pm}\right) \le C\|B_0\|\|\bigtriangledown \eta_{n+1}^{\pm}\|\|\bigtriangledown \xi_{n+1}^{\pm}\| \\ \le \varepsilon\|\bigtriangledown \xi_{n+1}^{\pm}\|^2 + \frac{C^2}{4\varepsilon}\|B_0\|^2\|\bigtriangledown \eta_{n+1}^{\pm}\|^2, \tag{7.6}$$

$$b\left(D_2 z_{n+1}^{\mp}, z_{n+1}^{\pm},\xi_{n+1}^{\pm}\right) \le \varepsilon\|\bigtriangledown \xi_{n+1}^{\pm}\|^2 + \frac{C^2}{4\varepsilon}\|\bigtriangledown z_{n+1}^{\pm}\|^2\|\bigtriangledown D_2 z_{n+1}^{\mp}\|^2 \\ \le \varepsilon\|\bigtriangledown \xi_{n+1}^{\pm}\|^2 + \frac{C^2}{4\varepsilon}\|z^{\pm}\|_{\infty,1}^2\left(\Delta t^3\int_{t_{n-1}}^{t_{n+1}}\|\bigtriangledown z_{tt}^{\mp}\|^2\right), \tag{7.7}$$

$$\begin{aligned}
&b\left(2e_n^{\mp}, z_{n+1}^{\pm},\xi_{n+1}^{\pm}\right)\\
&= 2b\left(\eta_n^{\mp}, z_{n+1}^{\pm},\xi_{n+1}^{\pm}\right) + 2b\left(\xi_n^{\mp}, z_{n+1}^{\pm},\xi_{n+1}^{\pm}\right)\\
&\le C\|\bigtriangledown \eta_n^{\mp}\|\|\bigtriangledown z_{n+1}^{\pm}\|\|\bigtriangledown \xi_{n+1}^{\pm}\| + C\|\bigtriangledown \xi_n^{\mp}\|^{1/2}\|\xi_n^{\mp}\|^{1/2}\|\bigtriangledown z_{n+1}^{\pm}\|\|\bigtriangledown \xi_{n+1}^{\pm}\|\\
&\le \left(\varepsilon\|\bigtriangledown \xi_{n+1}^{\pm}\|^2 + \frac{C^2}{4\varepsilon}\|\bigtriangledown \eta_n^{\mp}\|^2\|\bigtriangledown z_{n+1}^{\pm}\|^2\right) + \left(\varepsilon\|\bigtriangledown \xi_{n+1}^{\pm}\|^2 + \frac{C^2}{4\varepsilon}\|\bigtriangledown \xi_n^{\mp}\|\|\xi_n^{\mp}\|\|\bigtriangledown z_{n+1}^{\pm}\|^2\right)\\
&\le \left(\varepsilon\|\bigtriangledown \xi_{n+1}^{\pm}\|^2 + \frac{C^2}{4\varepsilon}\|\bigtriangledown \eta_n^{\mp}\|^2\|\bigtriangledown z_{n+1}^{\pm}\|^2\right) + \left(\varepsilon\|\bigtriangledown \xi_{n+1}^{\pm}\|^2 + \varepsilon\|\bigtriangledown \xi_n^{\mp}\|^2 + \frac{C^4}{64\varepsilon^3}\|\xi_n^{\mp}\|^2\|\bigtriangledown z_{n+1}^{\pm}\|^4\right)\\
&\le 2\varepsilon\|\bigtriangledown \xi_{n+1}^{\pm}\|^2 + \varepsilon\|\bigtriangledown \xi_n^{\mp}\|^2 + \frac{C^2}{4\varepsilon}\|z^{\pm}\|_{\infty,1}^2\|\bigtriangledown \eta_n^{\mp}\|^2 + \frac{C^4}{64\varepsilon^3}\|z^{\pm}\|_{\infty,1}^4\|\xi_n^{\mp}\|^2,
\end{aligned} \tag{7.8}$$

$$\begin{aligned}
&b\left(e_{n-1}^{\mp}, z_{n+1}^{\pm},\xi_{n+1}^{\pm}\right)\\
&= b\left(\eta_{n-1}^{\mp}, z_{n+1}^{\pm},\xi_{n+1}^{\pm}\right) + b\left(\xi_{n-1}^{\mp}, z_{n+1}^{\pm},\xi_{n+1}^{\pm}\right)\\
&\le C\|\bigtriangledown \eta_{n-1}^{\mp}\|\|\bigtriangledown z_{n+1}^{\pm}\|\|\bigtriangledown \xi_{n+1}^{\pm}\| + C\|\bigtriangledown \xi_{n-1}^{\mp}\|^{1/2}\|\xi_{n-1}^{\mp}\|^{1/2}\|\bigtriangledown z_{n+1}^{\pm}\|\|\bigtriangledown \xi_{n+1}^{\pm}\|\\
&\le \left(\varepsilon\|\bigtriangledown \xi_{n+1}^{\pm}\|^2 + \frac{C^2}{4\varepsilon}\|\bigtriangledown \eta_{n-1}^{\mp}\|^2\|\bigtriangledown z_{n+1}^{\pm}\|^2\right) + \left(\varepsilon\|\bigtriangledown \xi_{n+1}^{\pm}\|^2 + \frac{C^2}{4\varepsilon}\|\bigtriangledown \xi_{n-1}^{\mp}\|\|\xi_{n-1}^{\mp}\|\|\bigtriangledown z_{n+1}^{\pm}\|^2\right)\\
&\le \left(\varepsilon\|\bigtriangledown \xi_{n+1}^{\pm}\|^2 + \frac{C^2}{4\varepsilon}\|\bigtriangledown \eta_{n-1}^{\mp}\|^2\|\bigtriangledown z_{n+1}^{\pm}\|^2\right) + \left(\varepsilon\|\bigtriangledown \xi_{n+1}^{\pm}\|^2 + \varepsilon\|\bigtriangledown \xi_{n-1}^{\mp}\|^2 + \frac{C^4}{64\varepsilon^3}\|\xi_{n-1}^{\mp}\|^2\|\bigtriangledown z_{n+1}^{\pm}\|^4\right)\\
&\le 2\varepsilon\|\bigtriangledown \xi_{n+1}^{\pm}\|^2 + \|\bigtriangledown \xi_{n-1}^{\mp}\|^2 + \frac{C^2}{4\varepsilon}\|z^{\pm}\|_{\infty,1}^2\|\bigtriangledown \eta_{n-1}^{\mp}\|^2 + \frac{C^4}{64\varepsilon^3}\|z^{\pm}\|_{\infty,1}^4\|\xi_{n-1}^{\mp}\|^2.
\end{aligned} \tag{7.9}$$

Using the a priori from the stability analysis, i.e., $\|z_{h,n}^{\mp}\| \le C$, we have

$$b\left(2z_{h,n}^{\mp}, e_{n+1}^{\pm},\xi_{n+1}^{\pm}\right) = 2b\left(z_{h,n}^{\mp},\eta_{n+1}^{\pm},\xi_{n+1}^{\pm}\right) \\ \le C\|\bigtriangledown z_{h,n}^{\mp}\|^{1/2}\|z_{h,n}^{\mp}\|^{1/2}\|\bigtriangledown \eta_{n+1}^{\pm}\|\|\bigtriangledown \xi_{n+1}^{\pm}\| \\ \le \varepsilon\|\bigtriangledown \xi_{n+1}^{\pm}\|^2 + \frac{C^2}{4\varepsilon}\|\bigtriangledown z_{h,n}^{\mp}\|\|z_{h,n}^{\mp}\|\|\bigtriangledown \eta_{n+1}^{\pm}\|^2 \\ \le \varepsilon\|\bigtriangledown \xi_{n+1}^{\pm}\|^2 + \frac{C}{4\varepsilon}\|\bigtriangledown z_{h,n}^{\mp}\|\|\bigtriangledown \eta_{n+1}^{\pm}\|^2, \tag{7.10}$$

and similarly

$$b\left(2z_{h,n-1}^{\mp}, e_{n+1}^{\pm},\xi_{n+1}^{\pm}\right) = 2b\left(z_{h,n-1}^{\mp},\eta_{n+1}^{\pm},\xi_{n+1}^{\pm}\right)$$

$$
\begin{aligned}
&\leq C\| \nabla z_{h,n-1}^{\mp}\|^{1/2}\|z_{h,n-1}^{\mp}\|^{1/2}\| \nabla \eta_{n+1}^{\pm}\|\| \nabla \xi_{n+1}^{\pm}\| \\
&\leq \varepsilon\| \nabla \xi_{n+1}^{\pm}\|^2 + \frac{C}{4\varepsilon}\| \nabla z_{h,n-1}^{\mp}\|\| \nabla \eta_{n+1}^{\pm}\|^2. \qquad (7.11)
\end{aligned}
$$

Next, we add and subtract $\nu^- a\left((2z_n^{\mp} - z_{n-1}^{\mp}), \xi_{n+1}^{\pm}\right)$ to the linear term $\mathcal{M}_{n+1}^{\pm}$ so that

$$
\mathcal{M}_{n+1}^{\pm} = \nu^+ a\left(\eta_{n+1}^{\pm}, \xi_{n+1}^{\pm}\right) + \nu^- a\left(D_2 z_{n+1}^{\mp}, \xi_{n+1}^{\pm}\right) + \nu^- a\left((2e_n^{\mp} - e_{n-1}^{\mp}), \xi_{n+1}^{\pm}\right).
$$

These terms will be estimated as follows.

$$
\nu^+ a\left(\eta_{n+1}^{\pm}, \xi_{n+1}^{\pm}\right) \leq \nu^+\| \nabla \eta_{n+1}^{\pm}\|\| \nabla \xi_{n+1}^{\pm}\| \leq \varepsilon\| \nabla \xi_{n+1}^{\pm}\|^2 + \frac{(\nu^+)^2}{4\varepsilon}\| \nabla \eta_{n+1}^{\pm}\|^2, \qquad (7.12)
$$

$$
\begin{aligned}
\nu^-\left(\nabla D_2 z_{n+1}^{\mp}, \nabla \xi_{n+1}^{\pm}\right) &\leq \varepsilon\| \nabla \xi_{n+1}^{\pm}\|^2 + \frac{|\nu^-|^2}{4\varepsilon}\| \nabla D_2 z_{n+1}^{\mp}\|^2 \\
&\leq \varepsilon\| \nabla \xi_{n+1}^{\pm}\|^2 + \frac{|\nu^-|^2}{4\varepsilon}\left(\Delta t^3 \int_{t_{n-1}}^{t_{n+1}} \| \nabla z_{tt}^{\mp}\|^2\right), \qquad (7.13)
\end{aligned}
$$

and

$$
\begin{aligned}
&\nu^- a\left((2e_n^{\mp} - e_{n-1}^{\mp}), \xi_{n+1}^{\pm}\right) \\
&= \nu^-\left(\nabla(2\eta_n^{\mp} - \eta_{n-1}^{\mp}), \nabla \xi_{n+1}^{\pm}\right) + \nu^-\left(\nabla(2\xi_n^{\mp} - \xi_{n-1}^{\mp}), \nabla \xi_{n+1}^{\pm}\right) \\
&\leq 2|\nu^-|\| \nabla \eta_n^{\mp}\|\| \nabla \xi_{n+1}^{\pm}\| + |\nu^-|\| \nabla \eta_{n-1}^{\mp}\|\| \nabla \xi_{n+1}^{\pm}\| + 2|\nu^-|\| \nabla \xi_n^{\mp}\|\| \nabla \xi_{n+1}^{\pm}\| + |\nu^-|\| \nabla \xi_{n-1}^{\mp}\|\| \nabla \xi_{n+1}^{\pm}\| \\
&\leq 2\varepsilon\| \nabla \xi_{n+1}^{\pm}\|^2 + \frac{|\nu^-|^2}{\varepsilon}\| \nabla \eta_n^{\mp}\|^2 + \frac{|\nu^-|^2}{4\varepsilon}\| \nabla \eta_{n-1}^{\mp}\|^2 \\
&+ |\nu^-|\| \nabla \xi_n^{\mp}\|^2 + |\nu^-|\| \nabla \xi_{n+1}^{\pm}\|^2 + \frac{|\nu^-|}{2}\| \nabla \xi_{n-1}^{\mp}\|^2 + \frac{|\nu^-|}{2}\| \nabla \xi_{n+1}^{\pm}\|^2. \qquad (7.14)
\end{aligned}
$$

∎

Combine (7.3)-(7.14) with $\varepsilon = (\nu^+ - 3|\nu^-|)/34$, then (7.2) becomes

$$
\begin{aligned}
&\frac{1}{4\Delta t}\left(\|\xi_{n+1}^{\pm}\|^2 + \|2\xi_{n+1}^{\pm} - \xi_n^{\pm}\|^2\right) - \frac{1}{4\Delta t}\left(\|\xi_n^{\pm}\|^2 + \|2\xi_n^{\pm} - \xi_{n-1}^{\pm}\|^2\right) \\
&+ \frac{1}{4\Delta t}\|D_2 \xi_{n+1}^{\pm}\|^2 + \left(\nu^+ - \frac{3}{2}|\nu^-| - \frac{15}{34}(\nu^+ - 3|\nu^-|)\right)\| \nabla \xi_{n+1}^{\pm}\|^2 \\
&- \left(|\nu^-| + \frac{1}{34}(\nu^+ - 3|\nu^-|)\right)\| \nabla \xi_n^{\pm}\|^2 - \left(\frac{|\nu^-|}{2} + \frac{1}{34}(\nu^+ - 3|\nu^-|)\right)\| \nabla \xi_{n-1}^{\pm}\|^2 \\
&\leq \frac{C^4}{64\varepsilon^3}\|z^{\pm}\|_{\infty,1}^4\left(\|\xi_n^{\mp}\|^2 + \|\xi_{n-1}^{\mp}\|^2\right) \\
&+ \frac{C^2}{4\varepsilon\Delta t}\int_{t_{n-1}}^{t_{n+1}} \|\eta_t^{\pm}\|^2 dt + \frac{C^2 \Delta t^3}{4\varepsilon}\int_{t_{n-1}}^{t_{n+1}} \|z_{ttt}^{\pm}\|^2 dt + \frac{C^2}{4\varepsilon}\|p_{n+1} - p_{h,n+1}^{\pm}\|^2 \\
&+ \frac{C^2}{4\varepsilon}\|B_0\|^2\| \nabla \eta_{n+1}^{\pm}\|^2 + \frac{C^2}{4\varepsilon}\|z^{\pm}\|_{\infty,1}^2\left(\Delta t^3 \int_{t_{n-1}}^{t_{n+1}} \| \nabla z_{tt}^{\mp}\|^2\right) + \frac{C^2}{4\varepsilon}\|z^{\pm}\|_{\infty,1}^2\| \nabla \eta_n^{\mp}\|^2 + \\
&+ \frac{C^2}{4\varepsilon}\|z^{\pm}\|_{\infty,1}^2\| \nabla \eta_{n-1}^{\mp}\|^2 + \frac{C}{4\varepsilon}\| \nabla z_{h,n}^{\mp}\|\| \nabla \eta_{n+1}^{\pm}\|^2 + \frac{C}{4\varepsilon}\| \nabla z_{h,n-1}^{\mp}\|\| \nabla \eta_{n+1}^{\pm}\|^2 \\
&+ \frac{(\nu^+)^2}{4\varepsilon}\| \nabla \eta_{n+1}^{\pm}\|^2 + \frac{|\nu^-|^2}{4\varepsilon}\left(\Delta t^3 \int_{t_{n-1}}^{t_{n+1}} \| \nabla z_{tt}^{\mp}\|^2\right) + \frac{|\nu^-|^2}{\varepsilon}\| \nabla \eta_n^{\mp}\|^2 + \frac{|\nu^-|^2}{4\varepsilon}\| \nabla \eta_{n-1}^{\mp}\|^2. \qquad (7.15)
\end{aligned}
$$

We then sum up (7.15) from $n=1$ to $n=N-1$ and multiply by $2\Delta t$. Applying the Gronwall inequality results in

$$
\begin{aligned}
&\frac{1}{2}\left(\|\xi_N^{\pm}\|^2+\|2\xi_N^{\pm}-\xi_{N-1}^{\pm}\|^2\right)+\frac{1}{2}\sum_{n=1}^{N-1}\|D_2\xi_{n+1}^{\pm}\|^2+\Delta t(\nu^+-3|\nu^-|)\sum_{n=2}^{N}\|\bigtriangledown\xi_n^{\pm}\|^2\\
&\leq \exp(CN\Delta t)\Bigg\{\frac{1}{2}\left(\|\xi_1^{\pm}\|^2+\|2\xi_1^{\pm}-\xi_0^{+}\|^2\right)+2\Delta t\nu^+\left(\|\bigtriangledown\xi_1^{\pm}\|^2+\|\bigtriangledown\xi_0^{\pm}\|^2\right)\\
&+\frac{C^2}{2\varepsilon}\sum_{n=1}^{N-1}\int_{t_{n-1}}^{t_{n+1}}\|\eta_t^{\pm}\|^2dt+\frac{C^2\Delta t^4}{2\varepsilon}\sum_{n=1}^{N-1}\int_{t_{n-1}}^{t_{n+1}}\|z_{ttt}^{\pm}\|^2dt\\
&+\frac{C^2\Delta t}{2\varepsilon}\sum_{n=1}^{N-1}\|p_{n+1}-p_{h,n+1}^{\pm}\|^2+\frac{C^2\Delta t}{2\varepsilon}\|B_0\|^2\sum_{n=1}^{N-1}\|\bigtriangledown\eta_{n+1}^{\pm}\|^2\\
&+\frac{C^2\Delta t^4}{2\varepsilon}\|z^{\pm}\|_{\infty,1}^2\sum_{n=1}^{N-1}\int_{t_{n-1}}^{t_{n+1}}\|\bigtriangledown z_{tt}^{\mp}\|^2+\frac{C^2\Delta t}{2\varepsilon}\|z^{\pm}\|_{\infty,1}^2\sum_{n=1}^{N-1}\|\bigtriangledown\eta_n^{\mp}\|^2\\
&+\frac{C^2\Delta t}{2\varepsilon}\|z^{\pm}\|_{\infty,1}^2\sum_{n=1}^{N-1}\|\bigtriangledown\eta_{n-1}^{\mp}\|^2+\frac{C\Delta t}{2\varepsilon}\sum_{n=1}^{N-1}\|\bigtriangledown z_{h,n}^{\mp}\|\|\bigtriangledown\eta_{n+1}^{\pm}\|^2\\
&+\frac{C\Delta t}{2\varepsilon}\sum_{n=1}^{N-1}\|\bigtriangledown z_{h,n-1}^{\mp}\|\|\bigtriangledown\eta_{n+1}^{\pm}\|^2+\frac{(\nu^+)^2\Delta t}{2\varepsilon}\sum_{n=1}^{N-1}\|\bigtriangledown\eta_{n+1}^{\pm}\|^2\\
&+\frac{|\nu^-|^2\Delta t^4}{2\varepsilon}\sum_{n=1}^{N-1}\int_{t_{n-1}}^{t_{n+1}}\|\bigtriangledown z_{tt}^{\mp}\|^2+\frac{2|\nu^-|^2\Delta t}{\varepsilon}\sum_{n=1}^{N-1}\|\bigtriangledown\eta_n^{\mp}\|^2+\frac{|\nu^-|^2\Delta t}{2\varepsilon}\sum_{n=1}^{N-1}\|\bigtriangledown\eta_{n-1}^{\mp}\|^2\Bigg\}.\\
&\leq C_1\exp(CN\Delta t)\Bigg\{\|z_1^{\pm}-z_{h,1}^{\pm}\|^2+\|z_0^{\pm}-z_{h,0}^{\pm}\|^2+\|\bigtriangledown(z_1^{\pm}-z_{h,1}^{\pm})\|^2+\|\bigtriangledown(z_0^{\pm}-z_{h,0}^{\pm})\|^2\\
&+h^{2k+2}|||z^{\pm}|||_{\infty,k+1}^2+h^{2k+2}\|z_t^{\pm}\|_{2,k+1}^2+\Delta t^4\|z_{ttt}^{\pm}\|_{2,0}^2+h^{2s+2}|||p|||_{2,s+1}^2\\
&+h^{2k}|||z^{\pm}|||_{2,k+1}^2+\Delta t^4\|\bigtriangledown z_{tt}^{\mp}\|_{2,0}^2+h^{2k}|||z^{\pm}|||_{4,k+1}^4\Bigg\},
\end{aligned}
$$

where we used the standard interpolation error estimates. As a consequence, there exists a positive constant C_0 such that

$$
\begin{aligned}
&\frac{1}{2}\left(\|\xi_N^{+}\|^2+\|\xi_N^{-}\|^2+\|2\xi_N^{+}-\xi_{N-1}^{+}\|^2+\|2\xi_N^{-}-\xi_{N-1}^{-}\|^2\right)\\
&+\Delta t(\nu^+-3|\nu^-|)\sum_{n=2}^{N}\left(\|\bigtriangledown\xi_n^{+}\|^2+\|\bigtriangledown\xi_n^{-}\|^2\right)+\frac{1}{2}\sum_{n=1}^{N-1}\left(\|\delta\xi_{n+1}^{+}\|^2+\|\delta\xi_{n+1}^{-}\|^2\right)\\
&\leq C_0\Bigg\{\|z_1^{+}-z_{h,1}^{+}\|^2+\|z_1^{-}-z_{h,1}^{-}\|^2+\|z_0^{+}-z_{h,0}^{+}\|^2+\|z_0^{-}-z_{h,0}^{-}\|^2\\
&+\|\bigtriangledown(z_1^{+}-z_{h,1}^{+})\|^2+\|\bigtriangledown(z_1^{-}-z_{h,1}^{-})\|^2+\|\bigtriangledown(z_0^{+}-z_{h,0}^{+})\|^2+\|\bigtriangledown(z_0^{-}-z_{h,0}^{-})\|^2\\
&+h^{2k+2}|||z^{+}|||_{\infty,k+1}^2+h^{2k+2}|||z^{-}|||_{\infty,k+1}^2\\
&+\Delta t^4\|z_{ttt}^{+}\|_{2,0}^2++\Delta t^4\|z_{ttt}^{-}\|_{2,0}^2+\Delta t^4\|\bigtriangledown z_{tt}^{+}\|_{2,0}^2++\Delta t^4\|\bigtriangledown z_{tt}^{-}\|_{2,0}^2+h^{2s+2}|||p|||_{2,s+1}^2\\
&+h^{2k}|||z^{+}|||_{2,k+1}^2+h^{2k}|||z^{-}|||_{2,k+1}^2+h^{2k}|||z^{+}|||_{4,k+1}^4+h^{2k}|||z^{-}|||_{4,k+1}^4
\end{aligned}
\tag{7.16}
$$

$$+ h^{2k+2}\|z_t^+\|^2_{2,k+1}\} + h^{2k+2}\|z_t^-\|^2_{2,k+1}\Bigg\}.$$

To complete the error estimates, we add both sides of (7.16) with

$$Extra\_terms = \frac{1}{2}\left(\|\eta_N^+\|^2 + \|\eta_N^-\|^2 + \|2\eta_N^+ - \eta_{N-1}^+\|^2 + \|2\eta_N^- - \eta_{N-1}^-\|^2\right)$$
$$+ \Delta t(\nu^+ - 3|\nu^-|)\sum_{n=2}^{N}\left(\|\bigtriangledown \eta_n^+\|^2 + \|\bigtriangledown \eta_n^-\|^2\right) + \frac{1}{2}\sum_{n=1}^{N-1}\left(\|D_2\eta_{n+1}^+\|^2 + \|D_2\eta_{n+1}^-\|^2\right),$$

and apply the triangle inequality for the left-hand side. Noticing that the upcoming new terms are already contained in the right hand side of (7.16), we conclude the proof.
□

Proof. [Proof of Theorem 4.2] In this case we would add and subtract $\nu^- a\left(z^{\mp}_{h,n+1}, \xi^{\pm}_{n+1}\right)$ to the linear term $\mathcal{M}^{\pm}_{n+1}$, which becomes

$$\mathcal{M}^{\pm}_{n+1} = \nu^+ a\left(\eta^{\pm}_{n+1}, \xi^{\pm}_{n+1}\right) + \nu^- a\left(e^{\mp}_{n+1}, \xi^{\pm}_{n+1}\right) + \nu^- a\left(D_2 z^{\mp}_{h,n+1}, \xi^{\pm}_{n+1}\right).$$

The last terms is bounded by

$$\begin{aligned}
&\nu^- a\left(D_2 z^{\mp}_{h,n+1}, \xi^{\pm}_{n+1}\right)\\
&= \nu^- a\left(D_2 z^{\mp}_{n+1} - D_2\eta^{\mp}_{n+1} - D_2\xi^{\mp}_{n+1}, \xi^{\pm}_{n+1}\right)\\
&\le |\nu^-|\Big(\|\nabla D_2 z^{\mp}_{n+1}\|\|\nabla\xi^{\pm}_{n+1}\| + \|\nabla D_2\eta^{\mp}_{n+1}\|\|\nabla\xi^{\pm}_{n+1}\| + \|\nabla D_2\xi^{\mp}_{n+1}\|\|\nabla\xi^{\pm}_{n+1}\|\Big)\\
&\le 3\varepsilon\|\nabla\xi^{\pm}_{n+1}\|^2 + \frac{|\nu^-|^2}{4\varepsilon}\Big(\|\nabla D_2 z^{\mp}_{n+1}\|^2 + \|\nabla D_2\eta^{\mp}_{n+1}\|^2 + \|\nabla D_2\xi^{\mp}_{n+1}\|^2\Big)\\
&\le 3\varepsilon\|\nabla\xi^{\pm}_{n+1}\|^2 + \frac{|\nu^-|^2}{4\varepsilon}\Big(\|\nabla D_2 z^{\mp}_{n+1}\|^2 + \|\nabla D_2\eta^{\mp}_{n+1}\|^2\Big) + \frac{|\nu^-|^2 C^2_{INV}}{4\varepsilon h^2}\|\nabla D_2\xi^{\mp}_{n+1}\|^2,
\end{aligned}$$

where we used the inverse inequality at the last step. Treating all the other terms analogously as in the previous proof, we obtain (4.3). □

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