Problem 1 Let $A$, $B$ be complex square matrices with minimal polynomials $m_A(s)$, $m_B(s)$ and characteristic polynomials $p_A(s)$, $p_B(s)$. Show that if $m_A(s) = p_B(s)$ and $m_B(s) = p_A(s)$, then $A$ and $B$ are similar.

Problem 2 Let $A_n$ be a $2^n \times 2^n$ real matrix defined recursively by

\[
A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
A_n = \begin{pmatrix} A_{n-1} & I_{2^{n-1}} \\ I_{2^{n-1}} & -A_{n-1} \end{pmatrix} \text{ for } n \geq 2.
\]

Find $\text{det} A_n$ and justify your answer. (Hint: Calculate $A_2^n$.)

Problem 3 Let $A$ be a real $n \times n$ matrix. Suppose that the symmetric matrix $A + A^T$ has eigenvalues $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$. Show that any eigenvalue $\lambda$ of $A$ satisfies

\[
\frac{\mu_1}{2} \leq \text{Re} \lambda \leq \frac{\mu_n}{2}.
\]

Problem 4 Let $M_3$ be the collection of all $3 \times 3$ complex matrices with the natural linear structure. Let $T : M_3 \to M_3$ be the linear map

\[
T \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{21} + a_{12} & a_{13} & 0 \\ a_{31} & a_{32} & a_{33} \\ 0 & a_{32} + a_{23} \end{pmatrix}.
\]

(a) Show that $T$ is Nilpotent, i.e., $T^k = 0$ for some $k \geq 1$.

(b) Find the Jordan Canonical Form of $T$. (Hint: The JCF should be a $9 \times 9$ matrix.)

Problem 5 Let $A$ be an anti-selfadjoint map on a finite-dimensional complex Euclidean space. Show that

(a) $A - I$ is invertible.

(b) If $U = (A + I)(A - I)^{-1}$, then $U$ is unitary and $U - I$ is invertible.

Problem 6 (a) Let $x = (x_1, \ldots, x_n)^T$, $y = (y_1, \ldots, y_n)^T$ be two column vectors in $\mathbb{R}^n$, $n \geq 2$, such that $|x| = |y| = 1$ and $x \perp y$. Show that

\[
x_1^2 + y_1^2 \leq 1.
\]

(b) Let $A$ be a real $n \times n$ symmetric matrix with $n$ eigenvalues $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$. Show that for any $x, y \in \mathbb{R}^n$ such that $|x| = |y| = 1$ and $x \perp y$,

\[
|(x, Ax) + (y, Ay)| \leq |\lambda_1| + |\lambda_2|.
\]