Linear Algebra Preliminary Exam

May 11, 2021

1. Let A be an $n \times n$ matrix with complex entries. Prove that A is the sum of two non-singular matrices.

2. Let $n \in \mathbb{N} \cup \{0\}$ and V be the vector space of all polynomials in x with real coefficients and degree at most n. Let $P(x) \in V$ such that $\deg(P) = n$. Prove that
$$P(x),\ \frac{dP(x)}{dx},\ \ldots,\ \frac{d^n P(x)}{dx^n}$$
form a basis for V.

3. Let V be an n-dimensional vector space. Suppose that $T \in L(V, V)$ has a cyclic vector, i.e., there is a vector $x \in V$ such that $x, Tx, \cdots, T^{n-1}x$ form a basis of V. Show that if $U \in L(V, V)$ and $UT = TU$, then U is a polynomial in T.

4. Let $n \geq 2$ and A be a complex $n \times n$ matrix. Suppose that A has a dominant eigenvalue λ in the sense that λ is a simple eigenvalue of A and any other eigenvalue μ of A satisfies $|\mu| < |\lambda|$. Show that
$$\lim_{k \to \infty} \frac{1}{\lambda^k} A^k$$
exists and the limit is a rank one matrix.

5. Let X be an $n \times n$ complex Euclidean space and P_1, P_2 be two orthogonal projections such that
$$P_1 + P_2 \leq I.$$
Show that $P_1 P_2 = P_2 P_1 = 0$.

6. Let A, B be two $n \times n$ complex matrices such that $A \geq 0$ and $AB + BA = 0$. Show that $AB = BA = 0$.