ANALYSIS OF THE RAW FILTER IN COMPOSITE-TENDENCY LEAPFROG INTEGRATIONS

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Abstract. The leapfrog time-differencing scheme is widely used in numerical models of the atmosphere and ocean. Its accuracy can be improved by replacing the stabilizing Robert–Asselin filter with the Robert–Asselin–Williams (RAW) filter and also by using a composite tendency, which is constructed as a linear combination of the filtered and unfiltered tendencies. This paper conducts a detailed, rigorous analysis of the RAW-filtered composite-tendency leapfrog scheme. To facilitate the analysis, the scheme is first reduced to an equivalent partitioned multistep method. Taylor expansions show that the overall truncation error can be of order 1, 2, 3, or 4, depending on the parameter values. Root locus curves are used to derive the stability domain analytically. Numerical tests show that the analytic stability limits are sharp. Numerical tests also demonstrate the importance of proper initialization of the provisional (unfiltered and once-filtered) values, if the high accuracy is not to be degraded.

1. Introduction. The second-order, centred time-differencing scheme is known in the atmospheric and oceanic sciences as the leapfrog scheme. This scheme is used in many current numerical models (e.g., [4, 6, 7, 9, 14]). In order to damp the computational mode of the leapfrog scheme, it is usual to apply a Robert–Asselin filter [1]. Unfortunately, this filter damps the physical mode as well as the computational mode, and it reduces the accuracy of the scheme to first-order. Williams [13, 15] proposed a modification to the Robert–Asselin filter, in order to eliminate the first-order truncation errors. The modified filter has subsequently been adopted as the default scheme in various models (e.g., [12, 17]). As examples of its impacts, the modification has been found to improve the skill of medium-range weather forecasts [2] and to improve the simulation of major oceanic current systems including the Gulf Stream [18]. The modified filter has become known as the Robert–Asselin–Williams (RAW) filter.

More recently, Williams [16] proposed two methods for further improving the RAW-filtered leapfrog scheme. In both methods, the improvements are to the order of the amplitude errors of oscillatory modes; the order of the phase errors is unaffected. The first improvement is that leapfrogging over a particular linear combination of the filtered and unfiltered tendencies can eliminate the leading third-order amplitude errors and produce fifth-order amplitude accuracy. The second improvement is that the use of a $(1, -4, 6, -4, 1)$ filter instead of a $(1, -2, 1)$ filter can eliminate the leading fifth-order amplitude errors and produce seventh-order amplitude accuracy. In other related work, the proposed use of a linear combination of tendencies has been analysed in semi-implicit integrations [3] and a higher-order Robert–Asselin-type time filter has been proposed and analyzed [11].

In the present paper, we focus on the first of the above two improvements, namely the composite-tendency scheme. We investigate the accuracy and the stability properties of this scheme, more rigorously and in more detail than has been done before. The key step in our analysis is to derive the equivalent partitioned multistep method. Using this result, we obtain the consistency order and stability domain of the scheme. This knowledge allows us to gain new insights into the relationships between the consistency order and the filter parameters. In addition, the equivalent partitioned multistep method yields a new relationship that must be satisfied by the initial condition, if the accuracy is not to be degraded by the initialization.

The paper is organized as follows. The proposed scheme is analyzed in Section 2, where the equivalent partitioned multistep method, the order of convergence, and the stability domain are derived. Section 3 presents simple numerical tests to verify the consistency order and the stability. The summary and discussion appear in Section 4.

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2. Linear analysis. This section presents a linear analysis of the RAW-filtered composite-tendency leapfrog scheme. In Section 2.1, the defining equations for the numerical scheme are stated. The equivalent partitioned method is derived in Theorem 2.1. Sections 2.2 and 2.3, respectively, derive the consistency order and the stability domain.

2.1. The numerical scheme. Applied to the complex oscillation equation,

$$\frac{du}{dt} = i\omega u, \ \omega \in \mathbb{R}, \tag{2.1}$$

the RAW-filtered composite-tendency leapfrog scheme ((5)–(7) in [16]) is given by

$$w^{n+1} = u^{n-1} + 2i\omega\Delta t(\gamma v^n + (1-\gamma)w^n), \tag{2.2}$$

$$u^n = v^n + \frac{\nu\alpha}{2}(w^{n+1} - 2v^n + u^{n-1}), \tag{2.3}$$

$$v^{n+1} = w^{n+1} + \frac{\nu(\alpha-1)}{2}(w^{n+1} - 2v^n + u^{n-1}), \tag{2.4}$$

where Δt denotes the time step. The three dimensionless parameters in the scheme are ν, α, and γ, where ν corresponds to the classical Robert–Asselin filter parameter, α partitions the RAW filter displacements between the n'th and $(n+1)$'th time levels, and γ specifies the weighting coefficients for the composite tendency. Although previous work [16] has assumed $0 \le \gamma \le 1$, which may be a natural choice on some grounds, here we allow γ to vary outside this range. The variables w^n, v^n, and u^n denote, respectively, the unfiltered, once-filtered, and twice-filtered values at time $t^n = n\Delta t$. The scheme recovers to the original RAW-filtered leapfrog scheme [13] when $\gamma = 1$, and recovers to the classical Robert–Asselin-filtered leapfrog scheme [1] when $\alpha = 1$.*

The key step to be achieved before performing the consistency and stability analysis is to derive the equivalent partitioned method. This step is achieved in Theorem 2.1.

THEOREM 2.1. *The system (2.2)-(2.4) is equivalent to the following partitioned multistep method:*

$$\begin{aligned} &u^{n+1} - \nu u^n - (1-\nu)u^{n-1} \\ =&i\omega\Delta t\Big((2 + \nu\gamma(\alpha-1))u^n + \nu(2\gamma + \alpha - 2 - 2\alpha\gamma)u^{n-1} + \nu(1-\alpha)(1-\gamma)u^{n-2}\Big), \end{aligned} \tag{2.5}$$

or, equivalently,

$$\begin{aligned} u^{n+1} - \nu u^n - (1-\nu)u^{n-1} =&i\omega\Delta t\Big([2 + \nu(\alpha-1)]u^n - \nu\alpha u^{n-1}\Big) \\ &+ i\omega\Delta t\nu(1-\alpha)(1-\gamma)\Big(u^n - 2u^{n-1} + u^{n-2}\Big). \end{aligned}$$

Proof. The proof involves expresssing w^{n+1} and v^n in terms of values of u, and then substituting them into (2.3) to obtain a single equation in u.

Step 1. Eliminating w^{n+1} from (2.2) and (2.3) gives

$$u^n = \nu\alpha u^{n-1} + \big(i\omega\Delta t\nu\alpha\gamma + (1-\nu\alpha)\big)v^n + i\omega\Delta t\nu\alpha(1-\gamma)w^n,$$

from which it follows that

$$w^n = p_1 u^n - p_2 u^{n-1} - p_3 v^n, \tag{2.6}$$

*Note that $\gamma = 1$ is not required for the scheme to recover to the classical Robert–Asselin-filtered leapfrog scheme, since $v^n = w^n$ in (2.4) if $\alpha = 1$.

where

$$\begin{aligned} p &= i\omega\Delta t\nu\alpha(1-\gamma), \\ p_1 &= 1/p, \\ p_2 &= \nu\alpha/p, \\ p_3 &= \big(i\omega\Delta t\nu\alpha\gamma + (1-\nu\alpha)\big)/p. \end{aligned}$$

Equivalently, we have

$$w^{n+1} = p_1 u^{n+1} - p_2 u^n - p_3 v^{n+1}. \tag{2.7}$$

Step 2. Eliminating v^n from (2.3) and (2.4) gives

$$-\left(1-\nu(\alpha+1)/2\right) w^{n+1} + \nu(\alpha-1)u^n + (1-\nu\alpha)v^{n+1} = \frac{\nu(\alpha-1)}{2}u^{n-1}. \tag{2.8}$$

By substituting (2.6) into (2.8), we have

$$-\left(1-\nu(\alpha+1)/2\right)\left(p_1 u^{n+1} - p_2 u^n - p_3 v^{n+1}\right) + \nu(\alpha-1)u^n + (1-\nu\alpha)v^{n+1} = \frac{\nu(\alpha-1)}{2}u^{n-1}.$$

Then v^{n+1} is found to be

$$v^{n+1} = q_1 u^{n+1} - q_2 u^n + q_3 u^{n-1}, \tag{2.9}$$

where

$$\begin{aligned} q &= \left(1-\nu(\alpha+1)/2\right) p_3 + (1-\nu\alpha), \\ q_1 &= \left(1-\nu(\alpha+1)/2\right) p_1/q, \\ q_2 &= \left(\left(1-\nu(\alpha+1)/2\right) p_2 + \nu(\alpha-1)\right)/q, \\ q_3 &= \nu(\alpha-1)/(2q). \end{aligned}$$

Equivalently, we have

$$v^n = q_1 u^n - q_2 u^{n-1} + q_3 u^{n-2}. \tag{2.10}$$

Step 3. Substitution of v^{n+1} into (2.7) gives

$$\begin{aligned} w^{n+1} =& p_1 u^{n+1} - p_2 u^n - p_3 v^{n+1} \\ =& p_1 u^{n+1} - p_2 u^n - p_3(q_1 u^{n+1} - q_2 u^n + q_3 u^{n-1}) \\ =& (p_1 - p_3 q_1)u^{n+1} + (p_3 q_2 - p_2)u^n - p_3 q_3 u^{n-1}. \end{aligned} \tag{2.11}$$

Finally, substitution of (2.10) and (2.11) into (2.3) completes the proof.

□

Having concluded the proof, we can now use the theorem to derive the consistency order and stability domain.

2.2. The consistency order. By Taylor expansions of $u(t^{n+1})$, $u(t^{n-1})$, and $u(t^{n-2})$ at time t^n, the truncation error of (2.5) is found to be

$$\tau_n(\Delta t) = \frac{1}{\Delta t}\left(u(t^{n+1}) - \nu u(t^n) - (1-\nu)u(t^{n-1})\right)$$

$$
\begin{aligned}
&- i\omega\left((2+\nu\gamma(\alpha-1))u(t^n)+\nu(2\gamma+\alpha-2-2\alpha\gamma)u(t^{n-1})+\nu(1-\alpha)(1-\gamma)u(t^{n-2})\right)\\
&=\left(\frac{1}{2}-\alpha\right)\nu(i\omega\Delta t)u'(t^n)+\frac{1}{6}\left(2-\nu(7-9\alpha)+6\nu\gamma(1-\alpha)\right)(i\omega\Delta t)^2u'(t^n)\\
&\qquad+\frac{\nu}{24}(25-28\alpha-24\gamma+24\alpha\gamma)(i\omega\Delta t)^3u'(t^n)+\mathcal{O}(\Delta t^4).
\end{aligned}
$$

We observe that the scheme is generally first order if $\nu\neq 0$.[†] The scheme becomes second order if $\alpha=1/2$, as noted by Williams [16]. A new result here is that the scheme becomes third order if $\alpha=1/2$ and $\gamma=(5\nu-4)/(6\nu)$. This third-order scheme would require $\gamma<0$ if $\nu<4/5$, but there is no reason to forbid negative values of γ. Finally, the scheme becomes fourth order if $\alpha=1/2,\nu=-8$, and $\gamma=11/12$, but this case is not of practical interest because the negative value of ν means the computational mode is amplified by the filter.

Durran [5] proves a general proposition relating the order of the overall truncation error of a time-stepping scheme to the order of the phase and amplitude errors. The above findings are consistent with this proposition. In particular, the proposition shows that schemes of order 2 have phase errors of exactly second order, but that schemes of order 4 have phase errors of exactly fourth order. Williams [16] reported fourth-order phase errors for the case $\alpha=1/2$, $\nu=-8$, $\gamma=(3-\nu)/(4-\nu)=11/12$, which is consistent with fourth-order accuracy according to the above results.

2.3. The stability region. The root locus curve of a multistep method is the oriented boundary of its stability region. Thus, one can find the stability region by determining the root locus curve, and such a method is called the root locus curve method (see e.g. [8] for more details). In this section, we determine the stability of the proposed schemes using the root locus curve method. We focus on the case $\alpha=1/2$, which is the composite-tendency RAW-filtered leapfrog scheme that is at least second order. We note that the stability of the case $\alpha=1$ has been determined in [10] as

$$
\omega\Delta t\leq\sqrt{\frac{2-\nu}{2+\nu}}.
$$

When $\alpha=1/2$, the characteristic equation of (2.5) is

$$
\zeta^3-\nu\zeta^2-(1-\nu)\zeta-z\left((2-\nu\gamma/2)\zeta^2+\nu(\gamma-3/2)\zeta+\nu(1-\gamma)/2\right)=0,
$$

where ζ denotes all points on the unit circle, i.e., $\zeta=e^{i\theta}$ for $\theta\in[0,2\pi]$, and $z\in\mathbb{C}$. The curve of z is the root locus curve. In (2.5), $z=i\omega\Delta t$ lies on the imaginary axis, and hence θ satisfies

$$
\cos\theta=1\ \text{ or }\ \cos\theta=\frac{1-(2-\nu)(1-\gamma)}{2(1-\gamma)},
$$

provided $\gamma\leq(3-\nu)/(4-\nu)$ to ensure $|\cos\theta|\leq 1$. Therefore,

$$
z=0\ \text{ or }\ z=\pm i\frac{2}{(1-\gamma)(4-\nu)}\sqrt{\frac{(3-\nu)-(4-\nu)\gamma}{1+\nu(1-\gamma)}}.
$$

These values indicate the intersections of the root locus curve with the imaginary axis in the complex plane. Thus, we learn that the scheme has a stable range of time steps provided that:

$$
\omega\Delta t\leq\frac{2}{(1-\gamma)(4-\nu)}\sqrt{\frac{(3-\nu)-(4-\nu)\gamma}{1+\nu(1-\gamma)}},\quad\gamma\leq(3-\nu)/(4-\nu).\tag{2.12}
$$

[†] If $\nu=0$, the scheme is generally second order, but then the filter is inactive and the computational mode is uncontrolled.

We note that the third-order scheme identified in Section 2.2 has a stable range for the small, positive values of ν that are used in practical applications.

The root locus curves for $\nu = 0.1$ and several combinations of α and γ are plotted in Figure 2.1. The stability region of the third-order scheme with $\gamma = (5\nu - 4)/(6\nu)$ is enclosed by the imaginary axis and the root locus curve on the left half of the plane, and the stability condition is $\omega\Delta t \leq 0.2929$. The stability regions of the rest of the schemes are enclosed by their root locus curves. When $\alpha = 1/2$, the root locus curves for $\gamma = (5\nu - 4)/(6\nu)$, $\gamma = 0$, and $\gamma = 1/2$ intersect with the imaginary axis at the three points derived above. In contrast, the root locus curves for $\gamma = (3 - \nu)/(4 - \nu)$ and $\gamma = 1$ intersect with the imaginary axis only at the origin, indicating that these schemes are unstable. As γ decreases starting from $\gamma = (3 - \nu)/(4 - \nu)$, i.e. as more of the unfiltered tendency is mixed into the composite tendency, the scheme becomes more stable in the sense that the finite stable range increases. In some cases (e.g., $\alpha = 0.5, \gamma = 0.5$) the scheme has a slightly greater finite stable range than the original Robert–Asselin scheme ($\alpha = 1$).

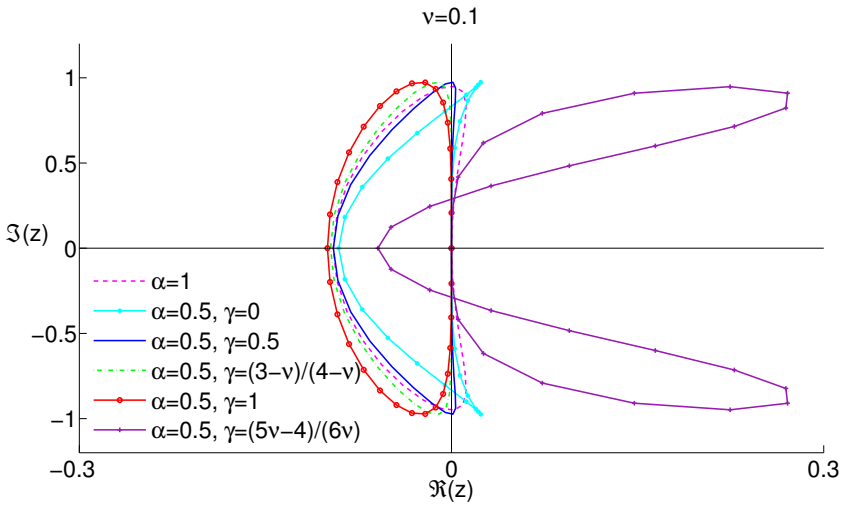

Fig. 2.1: Root locus curves for the scheme (2.2)-(2.4) are plotted for the case $\nu = 0.1$, for various combinations of α and γ. Note that $\alpha = 1$ corresponds to the original Robert–Asselin filter.

3. Numerical tests. This section presents some numerical tests on the composite-tendency RAW-filtered leapfrog scheme. Section 3.1 analyzes the consistency order and Section 3.2 analyzes the stability.

3.1. Numerical tests for the consistency order. Consider the oscillation equation (2.1). For simplicity, we take $\omega = 1$ and the initial condition $u(0) = 1$, for which the exact solution is $u(t) = e^{it}$. To test the impacts of initialization on the numerical errors, we now numerically integrate this oscillation equation using the scheme (2.2)–(2.4) over the time interval $[0, 10]$. The errors stated below are calculated as the absolute difference between the exact solution and the numerical solution at the final time, $t = 10$. We take $\nu = 0.1$, $\alpha = 1/2$, and $\gamma = (5\nu - 4)/(6\nu)$. For these parameters, the errors are expected to be $\mathcal{O}(\Delta t^3)$, as shown in Section 2.2.

For the first numerical experiment, the initialization method is as follows. Given $u^1 = u(0)$, the fourth-order Runge–Kutta (RK4) method is used to calculate u^2. We then take $w^2 = v^2 = u^2$ and start the integration loop from $n = 2$. The errors for various time steps are shown in Table 3.1, together with the inferred rates of convergence. We observe that the rate of convergence is somewhere between 2 and 3, which is not exactly the $\mathcal{O}(\Delta t^3)$ convergence suggested by Section 2.2.

Δt	Error	Rate
1/10	1.8960e-03	2.90
1/20	2.5461e-04	2.82
1/40	3.5998e-05	2.70
1/80	5.5301e-06	2.54
1/160	9.4822e-07	2.38
1/320	1.8275e-07	-

Table 3.1: The rate of convergence for the scheme (2.2)-(2.4) applied to the oscillation equation (2.1) for $\omega = 1$ and $u(0) = 1$. The parameters are $\alpha = 1/2$, $\nu = 0.1$, and $\gamma = (5\nu - 4)/(6\nu) \approx -5.8$. Given $u^1 = u(0)$, the RK4 method is used to calculate u^2, and then we take $w^2 = v^2 = u^2$, i.e., the initialization of w and v violates (2.6) and (2.10).

The reason for the apparent discrepancy is an improper initialization of the provisional values, v and w. We initialized w^2 and v^2 as $w^2 = v^2 = u^2$ and started the loop from $n = 2$. However, in order to recover strict third-order accuracy, w^n and v^n should depend on the u^n's following the relationships in (2.6) and (2.10). We now repeat the numerical experiment using an initialization method that satisfies these relationships, as follows. Given $u^1 = u(0)$, we use the RK4 method[‡] to calculate u^2 and u^3. We then calculate w^3 and v^3 according to (2.6) and (2.10) and start the integration loop from $n = 3$, i.e. the first loop computes w^4, u^3, and v^4. The results are shown in Table 3.2. We observe that the rate of convergence is now much closer to the $\mathcal{O}(\Delta t^3)$ convergence suggested by Section 2.2.

Δt	Error	Rate
1/10	1.7489e-03	2.99
1/20	2.2035e-04	2.99
1/40	2.7650e-05	3.00
1/80	3.4628e-06	3.00
1/160	4.3326e-07	3.00
1/320	5.4183e-08	-

Table 3.2: The rate of convergence for the scheme (2.2)-(2.4) applied to the oscillation equation (2.1) for $\omega = 1$ and $u(0) = 1$. The parameters are $\alpha = 1/2$, $\nu = 0.1$, and $\gamma = (5\nu - 4)/(6\nu) \approx -5.8$. Given $u^1 = u(0)$, the RK4 method is used to calculate u^2 and u^3, and then we calculate w^3 and v^3 according to (2.6) and (2.10).

3.2. Numerical tests for the stability. To test the analytic stability limits, we again consider the oscillation equation (2.1) with $\omega = 1$ and initial condition $u(0) = 1$. We numerically integrate this system using the scheme (2.2)-(2.4) with $\alpha = 1/2$, $\nu = 0.1$, and $\gamma = 0.5$. The initialization procedure used here is the same as that described in the caption to Table 3.2. According to (2.12), the maximum stable time step with which the oscillation equation can be integrated with these numerical parameters is $\delta t = 0.9756$. The energy of the numerical solution with time steps $\Delta t = 0.99\delta t$ and $\Delta t = 1.01\delta t$ over the time interval $[0, 200]$ are plotted in Figure 3.1. We observe that the maximum $\omega\Delta t$ derived in Section 2.3 is sharp, i.e. it cannot be made more restrictive.

[‡]Any method of at least third-order accuracy could be used.

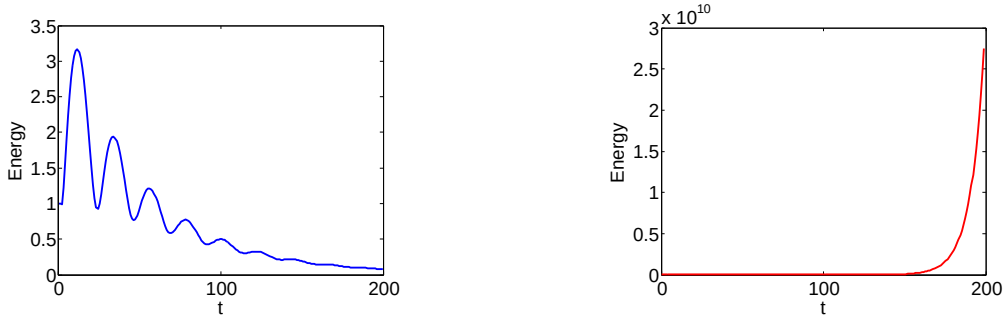

Fig. 3.1: Energy for the numerical solution of the scheme (2.2)-(2.4) with $\Delta t = 0.99\delta t$ (left) and $\Delta t = 1.01\delta t$ (right), where δt is the theoretical maximum stable time step derived in Section 2.3 for $\alpha = 1/2$, $\nu = 0.1$, and $\gamma = 0.5$.

4. Summary and discussion. In this paper, we have analyzed one of the schemes proposed by Williams in [16]. We have shown that the scheme applied to the complex oscillation equation is formally equivalent to a partitioned multistep method. We then used this result to derive analytically the consistency order (using Taylor expansions) and stability domain (using the root locus curve method). In general, when $\alpha = 1/2$, the scheme has $\mathcal{O}(\Delta t^2)$ truncation error. The truncation error increases to $\mathcal{O}(\Delta t^3)$ if, in addition to $\alpha = 1/2$, we also have $\gamma = (5\nu - 4)/(6\nu)$. We also numerically verified the order of convergence and stability. We have shown that care must be taken to avoid improper initialization of the unfiltered provisional value, w, and once-filtered provisional value, v, because increased truncation errors may result.

An important new result in this paper is the identification of the third-order scheme with $\alpha = 1/2$ and $\gamma = (5\nu - 4)/(6\nu)$. The identification of this scheme was facilitated by the partitioned multistep method. The scheme has a finite stable range of time steps and matches the accuracy of, for example, the third-order Adams–Bashforth method. Therefore, the scheme may be of practical interest in atmospheric and oceanic models. For values of ν in the practical range, γ will be negative. The form of (2.2) lends itself to an interpretation of γ as a parameter specifying the weighting coefficients of the unfiltered and singly filtered tendencies, and hence satisfying $0 \leq \gamma \leq 1$. However, re-writing (2.2) as

$$w^{n+1} = u^{n-1} + 2i\omega\Delta t(w^n + \gamma(v^n - w^n)) \tag{4.1}$$

shows that γ need not be restricted to this range, if the tendency is interpreted as the unfiltered tendency (w) modified by a term proportional to the difference between the singly filtered and unfiltered tendencies (v and w). Therefore, we see no problem in using a negative value of γ in order to obtain third-order

accuracy.

The reduction of numerical schemes to partitioned multistep methods is a powerful analysis tool which, when combined with Taylor expansions and the root locus curve method, can yield useful insights into the order of accuracy and stability domain. Such analysis methods could also be applied to the $(1, -4, 6, -4, 1)$ filter proposed by Williams [16], which achieves up to seventh-order amplitude accuracy. We leave this analysis for future work.

REFERENCES

[1] R. Asselin, *Frequency filter for time integrations*, Monthly Weather Review 100 (1972) 487-490. 1, 2

[2] J. Amezcua, E. Kalnay and P. D. Williams, *The effects of the RAW filter on the climatology and forecast skill of the SPEEDY model*, Monthly Weather Review 139(2) (2011) 608–619. 1

[3] J. Amezcua and P. D. Williams, *The composite-tendency RAW filter in semi-implicit integrations*, Quarterly Journal of the Royal Meteorological Society, submitted (2014). 1

[4] P. Bartello, *A comparison of time discretization schemes for two-timescale problems in geophysical fluid dynamics*, Journal of Computational Physics, 179 (1) (2002) 268–285. 1

[5] D. R. Durran, *The Third-Order Adams–Bashforth Method: An Attractive Alternative to Leapfrog Time Differencing*, Monthly Weather Review, 119 (3) (1991) 702–720. 4

[6] K. Fraedrich, H. Jansen, E. Kirk, U. Luksch, and F. Lunkeit, *The Planet Simulator: Towards a user friendly model*, Meteorologische Zeitschrift, 14 (3) (2005) 299–304. 1

[7] S. M. Griffies, R. C. Pacanowski, M. Schmidt, and V. Balaji, *Tracer conservation with an explicit free surface method for z-coordinate ocean models*, Monthly Weather Review, 129 (2001) 1081–1098. 1

[8] E. Hairer and G. Wanner, *Solving ordinary differential equations. II*, vol. 14 of Springer Series in Computational Mathematics, Springer-Verlag, Berlin, 2010. Stiff and differential-algebraic problems, Second revised edition. 4

[9] P. Hartogh, A. S. Medvedev, T. Kuroda, R. Saito, G. Vilanueva, A. G. Feofilov, A. A Kutepov, and U. Berger, *Description and climatology of a new general circulation model of the Martian atmosphere*, Journal of Geophysical Research, 110 (2005) E11008. 1

[10] N. Hurl, W. Layton, Y. Li, C. Trenchea, *Stability analysis of the Crank-Nicolson-Leapfrog method with the Robert-Asselin-Williams time filter*, BIT Numerical Mathematics, submitted (2013). 4

[11] Y. Li and C. Trenchea, *A higher-order Robert–Asselin type time filter*, Journal of Computational Physics 259 (2014) 23–32. 1

[12] M. Watanabe, and Coauthors, *Improved climate simulation by MIROC5: Mean states, variability, and climate sensitivity*, Journal of Climate, 23 (2010) 6312–6335. 1

[13] P. D. Williams, *A Proposed Modification to the Robert-Asselin Time Filter*, Monthly Weather Review, 137 (2009) 2538–2546. 1, 2

[14] P. D. Williams, T. W. N. Haine, P. L. Read, S. R. Lewis, and Y. H. Yamazaki, *QUAGMIRE v1.3: A quasi-geostrophic model for investigating rotating fluids experiments*, Geoscientific Model Development 2 (2009) 13–32. 1

[15] P. D. Williams, *The RAW filter: an improvement to the Robert–Asselin filter in semi-implicit integrations*, Monthly Weather Review 139(6) (2011) 1996–2007. 1

[16] P. D. Williams, *Achieving seventh-order amplitude accuracy in leapfrog integrations*, Monthly Weather Review, 141 (9) (2013) 3037–3051. 1, 2, 4, 7, 8

[17] C.-C. Young, Y.-H. Tseng, M.-L. Shen, Y.-C. Liang, M.-H. Chen, and C.-H. Chien, *Software development of the TaIwan Multi-scale Community Ocean Model (TIMCOM)*, Environmental Modelling & Software, 38 (2012) 214–219. 1

[18] C.-C. Young, Y.-C. Liang, Y.-H. Tseng, and C.-H. Chow, *Characteristics of the RAW filtered leapfrog time-stepping scheme in the ocean general circulation model*, Monthly Weather Review, 142 (2013) 434–447. 1