Problem 1. (15 pts.) a) Use **Gauss-Jordan elimination** (reduced row echelon form) to solve the system of linear equations

$$\begin{cases} x + y + z + w = 2 \\ 2x + 2y + z - w = -1 \\ 3x + 3y + 2z = 1 \\ x + y - 2w = -3 \end{cases}$$

or explain why the system is inconsistent. If the system is consistent, write down the solution in a vector form. NO CREDIT will be given, if **any other method** is used.

Problem 1. (CONTINUED)
b) For the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & -1 \\ 3 & 3 & 2 & 0 \\ 1 & 1 & 0 & -2 \end{pmatrix},$$

determine if the vector $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 1 \\ -3 \end{bmatrix}$ is in $col(A)$. Explain.

Problem 2. (20 pts.) The matrix A is given by

$$A = \begin{pmatrix} 3 & -2 & 1 & 2 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

a) Find a basis of $col(A)$ and the dimension of $col(A)$.

b) Find a basis of $row(A)$ and the dimension of $row(A)$.

Problem 2. (CONTINUED)
c) Find a basis of $null(A)$ and the dimension of $null(A)$.

d) Find a basis and the dimension of the orthogonal complement $W^{\perp}$ of the subspace W given by
$W = span\{[3, -2, 1, 2],\ [-1, 1, 0, 0],\ [1, 0, 1, 1]\}$

Problem 3. (10 pts.) Find the standard matrix of the linear transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$, that rotates a vector clockwise by 60° about the origin, then projects it onto the $y-$axis and then reflects it about the $x-$axis.

Problem 4. (20 pts.) For each given matrix A, determine if A is diagonalizable or not. If it is diagonalizable, find an invertible matrix P and a diagonal matrix D such that $D = P^{-1}AP$.
(You DO NOT have to find P^{-1}).
If A is not diagonalizable, explain why.

a) $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Problem 4. (CONTINUED)

b) $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

Problem 5. (10 pts.) a) If $W = span\,\{\mathbf{x_1},\ \mathbf{x_2}\}$ is a subspace of $\mathbb{R}^4$,

where $\mathbf{x_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$ and $\mathbf{x_2} = \begin{bmatrix} 2 \\ -1 \\ 2 \\ -1 \end{bmatrix}$,

use the Gram-Schmidt process to construct an orthogonal basis of W.

b) Use your answer in part a) to find the orthogonal projection $\mathbf{proj}_W(\mathbf{v})$

of the vector $\mathbf{v} = \begin{bmatrix} 5 \\ 4 \\ -3 \\ -2 \end{bmatrix}$ onto W.

Problem 6. (25 pts.) Determine if each of the statements below is TRUE or FALSE. Circle your choice and give the explanation for your answer.

a) If A is a square 4×4 matrix with $det(A) \neq 0$, and the vector $\mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 9 \end{bmatrix}$, then the system $(A|\mathbf{v})$ must have exactly one solution

TRUE FALSE

Explanation:

b) If A is a square 4×4 matrix with $det(A) = 0$, and the vector $\mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 9 \end{bmatrix}$, then the system $(A|\mathbf{v})$ must have no solutions

TRUE FALSE

Explanation:

c) If a square $n \times n$ matrix A can be represented as a product of elementary matrices, then A must be diagonalizable.

TRUE FALSE

Explanation:

d) If three non-zero vectors in $\mathbb{R}^3$ are linearly dependent, then two of them must be parallel

TRUE FALSE

Explanation:

Problem 6. (CONTINUED)

e) If a 5×3 matrix A has $rank(A) = 3$, then the columns of A must be linearly independent.

TRUE FALSE

Explanation:

f) If λ is an eigenvalue of a square matrix A, and the algebraic multiplicity of λ is 1, then the geometric multiplicity of λ can be

(i) 0

TRUE FALSE

(ii) 1

TRUE FALSE

(iii) 2

TRUE FALSE

Explanation:

g) Any four non-zero vectors in $\mathbb{R}^4$ must form a basis of $\mathbb{R}^4$

TRUE FALSE

Explanation:

h) If a 3×3 matrix A has a row-echelon form $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, then A is invertible

TRUE FALSE

Explanation: