Sample Final Exam 5

Solutions

1. (10 points) Let

$$P(1,0,2), \quad Q(3,1,0), \quad R(0,2,1).$$

(a) Find a scalar equation of the plane that passes through the points P, Q, and R.

(b) Find a parametric equation for the line through P and Q.

(c) Find the distance from the origin to the plane found in part (a).

Brief solution:

$\overrightarrow{PQ} = \langle 2, 1, -2\rangle$, $\overrightarrow{PR} = \langle -1, 2, -1\rangle$. A normal vector is

$$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \langle 3, 4, 5\rangle.$$

Plane through P:

$$3(x-1) + 4(y-0) + 5(z-2) = 0 \implies 3x + 4y + 5z = 13.$$

Line through P and Q:

$$\vec{\mathsf{r}}(t) = \langle 1, 0, 2\rangle + t\langle 2, 1, -2\rangle, \quad \text{so} \quad x = 1 + 2t,\ y = t,\ z = 2 - 2t.$$

Distance from origin to $3x + 4y + 5z = 13$:

$$d = \frac{|3\cdot 0 + 4\cdot 0 + 5\cdot 0 - 13|}{\sqrt{3^2 + 4^2 + 5^2}} = \frac{13}{\sqrt{50}} = \frac{13}{5\sqrt{2}}.$$

2. (10 points) Find the curvature of the curve

$$\vec{\mathsf{r}}(t) = \langle t^2,\ 2t + 4,\ t^3 + 4\rangle$$

at the point $P = (1, 2, 3)$. Simplify your answer.

Brief solution: Point P corresponds to $t = -1$ since

$$\vec{\mathsf{r}}(-1) = \langle 1, 2, 3\rangle.$$

Compute

$$\vec{\mathsf{r}}'(t) = \langle 2t, 2, 3t^2\rangle, \quad \vec{\mathsf{r}}''(t) = \langle 2, 0, 6t\rangle.$$

At $t = -1$:

$$\vec{r}'(-1) = \langle -2, 2, 3\rangle, \quad \vec{r}''(-1) = \langle 2, 0, -6\rangle.$$

Then

$$\vec{r}'(-1) \times \vec{r}''(-1) = \langle -12, -6, -4\rangle, \quad \|\vec{r}'(-1) \times \vec{r}''(-1)\| = 14,$$
$$\|\vec{r}'(-1)\| = \sqrt{(-2)^2 + 2^2 + 3^2} = \sqrt{17}.$$

Curvature:

$$\kappa = \frac{\|\vec{r}' \times \vec{r}''\|}{\|\vec{r}'\|^3} = \frac{14}{(\sqrt{17})^3} = \frac{14}{17\sqrt{17}}.$$

3. (10 points) Find and classify all critical points of the function

$$f(x, y) = \tfrac{1}{2}x^4 - xy + y^2.$$

Brief solution:

$$f_x = 2x^3 - y, \quad f_y = -x + 2y.$$

Set $f_x = f_y = 0$:

$$2x^3 - y = 0, \quad -x + 2y = 0 \;\Rightarrow\; y = 2x^3,\; 2y = x.$$

Thus $2(2x^3) = x \Rightarrow 4x^3 - x = 0 \Rightarrow x(4x^2 - 1) = 0$, so

$$x = 0,\; \pm\tfrac{1}{2}, \quad y = 2x^3 :\; (0, 0),\; (\tfrac{1}{2}, \tfrac{1}{4}),\; (-\tfrac{1}{2}, -\tfrac{1}{4}).$$

Second derivatives:

$$f_{xx} = 6x^2, \quad f_{yy} = 2, \quad f_{xy} = -1.$$

Discriminant:

$$D = f_{xx}f_{yy} - f_{xy}^2 = 12x^2 - 1.$$

At $(0, 0)$: $D = -1 < 0 \Rightarrow$ saddle. At $x = \pm\frac{1}{2}$: $D = 12 \cdot \frac{1}{4} - 1 = 3 - 1 = 2 > 0$ and $f_{xx} = 6x^2 > 0 \Rightarrow$ both $(\frac{1}{2}, \frac{1}{4})$ and $(-\frac{1}{2}, -\frac{1}{4})$ are local minima.

4. (10 points) Find the absolute maximum and the absolute minimum values of the function

$$f(x, y, z) = x + y + 2z,$$

subject to the constraint

$$x^2 + y^2 + 3z^2 = 30.$$

Brief solution: Constraint: $g(x,y,z) = x^2 + y^2 + 3z^2 - 30 = 0$. Lagrange equations: $\nabla f = \lambda \nabla g$:

$$\langle 1,1,2\rangle = \lambda\langle 2x, 2y, 6z\rangle \;\Rightarrow\; x = \frac{1}{2\lambda},\; y = \frac{1}{2\lambda},\; z = \frac{1}{3\lambda}.$$

Plug into $x^2 + y^2 + 3z^2 = 30$:

$$\frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} + 3\cdot\frac{1}{9\lambda^2} = \frac{5}{6\lambda^2} = 30 \;\Rightarrow\; \lambda^2 = \frac{1}{36}.$$

So $\lambda = \pm\frac{1}{6}$.
If $\lambda = \frac{1}{6}$, then $(x,y,z) = (3,3,2)$ and

$$f(3,3,2) = 3+3+4 = 10.$$

If $\lambda = -\frac{1}{6}$, then $(x,y,z) = (-3,-3,-2)$ and

$$f(-3,-3,-2) = -3-3-4 = -10.$$

Thus absolute maximum 10 at $(3,3,2)$ and absolute minimum -10 at $(-3,-3,-2)$.

5. (10 points) A lamina with the density $\rho(x,y) = 2x - y$ occupies a parallelogram with vertices $A(0,0)$, $B(3,1)$, $C(4,3)$, and $D(1,2)$. Use the change of variables $x = 3u + v$, $y = u + 2v$ to find the mass of the lamina.

Brief solution: Jacobian:

$$J = \det\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} = 6 - 1 = 5, \qquad dx\,dy = 5\,du\,dv.$$

Compute (u,v) for the vertices using $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$. The inverse matrix has determinant 5, so

$$u = \frac{1}{5}(2x - y), \qquad v = \frac{1}{5}(-x + 3y).$$

Then

$$A(0,0) \mapsto (0,0),\; B(3,1) \mapsto (1,0),\; C(4,3) \mapsto (1,1),\; D(1,2) \mapsto (0,1),$$

so the (u,v)-region is the unit square $0 \le u \le 1$, $0 \le v \le 1$.
Density:

$$\rho(x,y) = 2x - y = 2(3u+v) - (u+2v) = 5u.$$

Mass:

$$m = \iint (2x-y)\,dx\,dy = \iint 5u\cdot 5\,du\,dv = 25\int_0^1\int_0^1 u\,dv\,du = 25\int_0^1 u\,du = \frac{25}{2}.$$

6. (10 points) Evaluate $\displaystyle\iiint\limits_E z\,dV$, where E is the hemispherical region that lies above the xy-plane and below the sphere $x^2+y^2+z^2=4$.

Brief solution: Use spherical coordinates: $0 \le \rho \le 2$, $0 \le \phi \le \frac{\pi}{2}$, $0 \le \theta \le 2\pi$, with

$$z = \rho\cos\phi, \quad dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$$

Then

$$\iiint_E z\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^2 (\rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \left(\int_0^{2\pi} d\theta\right)\left(\int_0^2 \rho^3\,d\rho\right)\left(\int_0^{\pi/2}\sin\phi\cos\phi\,d\phi\right)$$

Compute:

$$\int_0^{2\pi} d\theta = 2\pi, \quad \int_0^2 \rho^3\,d\rho = 4, \quad \int_0^{\pi/2}\sin\phi\cos\phi\,d\phi = \tfrac{1}{2}.$$

Thus

$$\iiint_E z\,dV = 2\pi\cdot 4\cdot\tfrac{1}{2} = 4\pi.$$

7. (10 points) Set up an iterated integral for the area of the part of the surface $z = x^2y$ that lies above the triangle on the xy-plane with vertices $(0,0)$, $(2,0)$, and $(4,2)$. **DO NOT EVALUATE.**

Brief solution: For $z = f(x,y) = x^2y$, surface area is

$$A = \iint_D \sqrt{1+f_x^2+f_y^2}\,dA, \quad f_x = 2xy,\ f_y = x^2,$$

so the integrand is

$$\sqrt{1+4x^2y^2+x^4}.$$

The triangle has edges

$$AB:\ y=0, \quad AC:\ y=\tfrac{1}{2}x, \quad BC:\ y=x-2.$$

A convenient description of D:

$$0\le x\le 2, \quad 0\le y\le \tfrac{1}{2}x; \qquad 2\le x\le 4, \quad x-2\le y\le \tfrac{1}{2}x.$$

Thus

$$A = \int_0^2\int_0^{x/2}\sqrt{1+4x^2y^2+x^4}\,dy\,dx + \int_2^4\int_{x-2}^{x/2}\sqrt{1+4x^2y^2+x^4}\,dy\,dx.$$

8. (10 points)

   (a) (5 pts) Determine if the vector field

$$\vec{F} = \langle (1+xy)e^{xy},\, x^2e^{xy} \rangle$$

   is conservative.

   (b) (5 pts) Evaluate

$$\int_C (1+xy)e^{xy}\,dx + x^2e^{xy}\,dy,$$

   where C is the part of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$.

Brief solution: Let $P(x,y) = (1+xy)e^{xy}$, $Q(x,y) = x^2e^{xy}$.

(a) Note that

$$f(x,y) = xe^{xy} \quad \Rightarrow \quad f_x = (1+xy)e^{xy} = P, \quad f_y = x^2e^{xy} = Q.$$

So $\vec{F} = \nabla f$ and is conservative on $\mathbb{R}^2$.

(b) By the Fundamental Theorem for line integrals,

$$\int_C \vec{F} \cdot d\vec{r} = f(1,1) - f(0,0) = (1 \cdot e^1) - (0 \cdot e^0) = e.$$

So the integral equals e.

9. (10 points) Evaluate $\oint_C \vec{\mathsf{F}} \cdot d\,\vec{\mathsf{r}}$, where

$$\vec{\mathsf{F}} = yz\,\vec{\mathsf{i}} + 2z\,\vec{\mathsf{j}} + 3y\,\vec{\mathsf{k}} = \langle yz,\, 2z,\, 3y \rangle,$$

and C is the curve of intersection of the plane $2x + z = 4$ and the cylinder $x^2 + y^2 = 9$, oriented *clockwise* as viewed from above.

Brief solution: Use Stokes' Theorem. Let S be the portion of the plane $2x + z = 4$ inside the cylinder, with orientation chosen so that C is clockwise from above (this corresponds to a downward-pointing normal).

First compute

$$\operatorname{curl} \vec{\mathsf{F}} = \nabla \times \langle yz, 2z, 3y \rangle = \langle 1,\, y,\, -z \rangle.$$

On the plane $z = 4 - 2x$, so on S:

$$\operatorname{curl} \vec{\mathsf{F}} = \langle 1,\, y,\, 2x - 4 \rangle.$$

Write the plane as $z = g(x, y) = 4-2x$. For $z = g(x, y)$, an upward normal is $\langle -g_x, -g_y, 1\rangle = \langle 2, 0, 1\rangle$, so a downward normal is $\langle -2, 0, -1\rangle$.

Thus

$$(\text{curl}\,\vec{\mathsf{F}}) \cdot \langle -2, 0, -1\rangle = -2 - (2x - 4) = 2 - 2x = 2(1 - x).$$

The projection D of S onto the xy-plane is the disk $x^2 + y^2 \le 9$. By Stokes,

$$\oint_C \vec{\mathsf{F}} \cdot d\vec{\mathsf{r}} = \iint_S (\text{curl}\,\vec{\mathsf{F}}) \cdot \mathbf{n}\, dS = \iint_D 2(1 - x)\, dA.$$

In polar coordinates $x = r\cos\theta$, $0 \le r \le 3$, $0 \le \theta \le 2\pi$:

$$\oint_C \vec{\mathsf{F}} \cdot d\vec{\mathsf{r}} = \int_0^{2\pi} \int_0^3 2(1 - r\cos\theta)\, r\, dr\, d\theta = \int_0^{2\pi} \big(9 - 18\cos\theta\big)\, d\theta = 18\pi.$$

10. (10 points) Calculate the flux of the vector field

$$\vec{\mathsf{F}} = \langle xe^y,\, xy - e^y,\, x(2 - x)\rangle$$

across the sphere $x^2 + y^2 + z^2 = 9$, oriented outward.

Brief solution: Use the Divergence Theorem. Let E be the solid ball $x^2 + y^2 + z^2 \le 9$. Then

$$\iint_S \vec{\mathsf{F}} \cdot d\vec{\mathsf{S}} = \iiint_E (\nabla \cdot \vec{\mathsf{F}})\, dV.$$

Compute

$$\nabla \cdot \vec{\mathsf{F}} = \frac{\partial}{\partial x}(xe^y) + \frac{\partial}{\partial y}(xy - e^y) + \frac{\partial}{\partial z}\big(x(2 - x)\big) = e^y + (x - e^y) + 0 = x.$$

So we need $\iiint_E x\, dV$. The ball is symmetric about the yz-plane, and x is an odd function in x, so the integral over the ball is 0:

$$\iiint_E x\, dV = 0.$$

Therefore the flux is 0.