Solutions

1. (5 pts) Determine $\cosh(a)$ if $\sinh(a) = 5$.

Formula! $\cosh^2(a)-\sinh^2(a)=1$, so $\cosh(a)=\sqrt{1+(5)^2}=\boxed{\sqrt{26}}$

2. (5 pts) Evaluate and simplify $\tanh(\ln(4))$.

$\tanh(x)=\frac{\sinh(x)}{\cosh(x)}=\frac{\frac{e^x-e^{-x}}{2}}{\frac{e^x+e^{-x}}{2}}=\frac{e^x-e^{-x}}{e^x+e^{-x}}$, so $\tanh(\ln(4))=\frac{e^{\ln(4)}-e^{-\ln(4)}}{e^{\ln(4)}+e^{-\ln(4)}}$

$=\frac{4-e^{\ln(\frac{1}{4})}}{4+e^{\ln(\frac{1}{4})}}=\frac{4-\frac{1}{4}}{4+\frac{1}{4}}=\frac{\frac{15}{4}}{\frac{17}{4}}=\boxed{\frac{15}{17}}$

3. (10 pts) For the parametric curve, $x(t) = 3\sin(t)$ and $y(t) = 8\cos(t)$, determine the equation of the tangent line when $t = \frac{\pi}{3}$.

$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}$. Note $\frac{dy}{dt}=-8\sin(t)$ and $\frac{dx}{dt}=3\cos(t)$. When $t=\frac{\pi}{3}$, get

$\frac{dy}{dx}=\frac{-8\sin(\frac{\pi}{3})}{3\cos(\frac{\pi}{3})}=\frac{-8\cdot\frac{\sqrt{3}}{2}}{3\cdot\frac{1}{2}}=\boxed{\frac{-8\sqrt{3}}{3}}$

Equation of tangent line is $y-y_1=m(x-x_1)$.

$x_1=x(\frac{\pi}{3})=3\sin(\frac{\pi}{3})=\boxed{3\cdot\frac{\sqrt{3}}{2}}$, $y_1=y(\frac{\pi}{3})=8\cos(\frac{\pi}{3})=8\cdot\frac{1}{2}=\boxed{4}$

So, $\boxed{y-4=\frac{-8\sqrt{3}}{3}\left(x-\frac{3\sqrt{3}}{2}\right)}$

4. (5 pts) If $g(x) = \int_2^{x^2} \sin^6(u)\,du$, determine $g'(x)$.

$\boxed{g'(x)=\sin^6(x^2)\cdot 2x-\sin^6(2)\cdot 0}$

5. (5 pts each) Differentiate the following functions. You do not need to simplify your answer.

(a) $f(x) = \ln(3^x + 2e^{-2x})$

$$\boxed{f'(x) = \frac{1}{3^x + 2e^{-2x}} \cdot \left(3^x \ln(3) + 2e^{-2x} \cdot (-2)\right)}$$

(b) $y = \dfrac{2\sin^3(x)}{1 + x^3}$ ↙ Quotient

$$\boxed{y' = \frac{[1+x^3] \cdot [2 \cdot 3\sin^2(x) \cdot \cos(x)] - [2\sin^3(x)] \cdot [3x^2]}{(1+x^3)^2}}$$

(c) $f(x) = \sqrt{1 - 4x^2}\,\arcsin(2x)$ ↙ Product

$$\boxed{f'(x) = \left[\sqrt{1-4x^2}\right] \cdot \left[\frac{1}{\sqrt{1-(2x)^2}} \cdot 2\right] + \left[\frac{1}{2\sqrt{1-4x^2}} \cdot -8x\right] \cdot \left[\arcsin(2x)\right]}$$

(d) $y = \dfrac{10}{(x^2 + 5x + 1)^2} = 10(x^2+5x+1)^{-2}$

$$\boxed{y' = -20(x^2+5x+1)^{-3} \cdot (2x+5)}$$

Product

(e) $y = (3x+1)^2\,(x^3 - 4x^2 + x - 7)^4$

$$\boxed{y' = \{[(3x+1)^2] \cdot [4(x^3-4x^2+x-7)^3 \cdot (3x^2-8x+1)] + [2(3x+1) \cdot 3] \cdot \{[(x^3-4x^2+x-7)^4]}$$

Super-exponential

(f) $y = (2+\cos(x))^x$

$$\ln(y) = \ln\left((2+\cos(x))^x\right) = x \cdot \ln(2+\cos(x))$$

$$\frac{1}{y} \cdot y' = [x] \cdot \left[\frac{\sin(x)}{2+\cos(x)}\right] + [1] \cdot \ln(2+\cos(x))$$

$$\boxed{y' = \left[\frac{x\sin(x)}{2+\cos(x)} + \ln(2+\cos(x))\right] \cdot \left((2+\cos(x))^x\right)}$$

6. For the curve $(1+y^2)^3 + (1+x^2)^3 + 4xy = 20$

(a) (10 pts) Determine $\dfrac{dy}{dx}$ Differentiate w.r.t. x:

$$3(1+y^2)^2 \cdot 2y\,\frac{dy}{dt} + 3(1+x^2)^2 \cdot 2x + 4\left(x \cdot \frac{dy}{dx} + 1 \cdot y\right) = 0$$

$$6y(1+y^2)^2\,\frac{dy}{dt} + 6x(1+x^2)^2 + 4x\,\frac{dy}{dx} + 4y = 0$$

$$\frac{dy}{dt}\left(6y(1+y^2)^2 + 4x\right) = -4y - 6x(1+x^2)^2$$

$$\boxed{\frac{dy}{dx} = \frac{-4y - 6x(1+x^2)^2}{6y(1+y^2)^2 + 4x}}$$

(b) (5 pts) Determine the equation of the tangent line at the point $(1, 1)$.

Equation: $y - y_1 = m(x - x_1)$. $x_1 = 1$, $y_1 = 1$, $m = \left.\dfrac{dy}{dx}\right|_{(1,1)}$, so

$$m = \frac{-4(1) - 6(1)(1+(1)^2)^2}{6(1)(1+(1)^2)^2 + 4(1)} = \frac{-4 - 6(2)^2}{6(2)^2 + 4} = \frac{-28}{28} = \boxed{-1}.$$

Thus, get $\boxed{y - 1 = -1(x-1)}$

7. (15 pts) Determine the limit (show all work)

olution 1: L'Hospital's Rule

(a) $\lim\limits_{x\to0} \dfrac{x+3\sin(2x)}{6\tan(x)} \to \dfrac{0}{0}$

$\overset{LH}{=} \lim\limits_{x\to0} \dfrac{1+6\cos(2x)}{6\sec^2(x)} = \dfrac{1+6(1)}{6(1)^2} = \boxed{\dfrac{7}{6}}$

Solution 2: Previous results: $\lim\limits_{x\to0} \dfrac{\sin(x)}{x} = \lim\limits_{x\to0} \dfrac{\tan(x)}{x} = 1.$

$= \lim\limits_{x\to0} \dfrac{x}{6\tan(x)} + \lim\limits_{x\to0} \dfrac{\sin(2x)}{2\tan(x)}$

$= \dfrac{1}{6} \lim\limits_{x\to0} \left(\dfrac{x}{\tan(x)}\right) + \lim\limits_{x\to0} \dfrac{\sin(2x)}{2x} \cdot \dfrac{x}{\tan(x)}$

$= \dfrac{1}{6} \cdot \dfrac{1}{\lim\limits_{x\to0} \dfrac{\tan(x)}{x}} + \lim\limits_{x\to0} \dfrac{\sin(2x)}{2x} \cdot \dfrac{1}{\lim\limits_{x\to0} \dfrac{\tan(x)}{x}} = \dfrac{1}{6} \cdot 1 + 1 \cdot 1 = \boxed{\dfrac{7}{6}}$

(b) $\lim\limits_{x\to0} (1+2x)^{3x} \to 1^0$ NOT INDETERMINATE, get $\boxed{1}$.

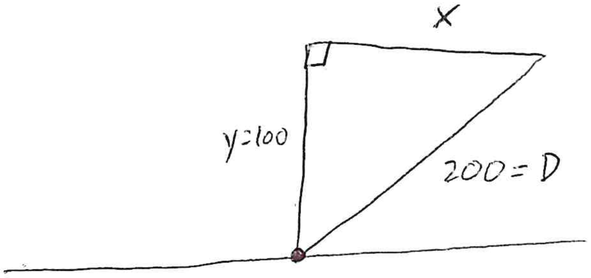
(Remark: We could write $L = \lim\limits_{x\to0} (1+2x)^{3x}$ and compute $\ln(L) = \lim\limits_{x\to0} \ln\left((1+2x)^{3x}\right)$

$= \lim\limits_{x\to0} 3x \cdot \ln(1+2x) \to 0 \cdot 0 = \boxed{0}$, then solve $L = e^0 = 1$.)

(c) $\lim\limits_{x\to\infty} (\ln(4x+1) - \ln(3x+2))$

$= \lim\limits_{x\to\infty} \ln\left(\dfrac{4x+1}{3x+2}\right) = \ln\left|\lim\limits_{x\to\infty} \dfrac{4x+1}{3x+2}\right) = \ln\left(\lim\limits_{x\to\infty} \dfrac{4+\frac{1}{x}}{3+\frac{2}{x}}\right) = \boxed{\ln\left(\frac{4}{3}\right)}$

8. (15 pts) A kite 100 ft above the ground moves horizontally at a rate of 8 ft/sec. At what rate is the angle between the string and the horizontal changing when 200 ft of string has been let out?



$$x^2+y^2=D^2.$$

Differentiate w.r.t. time:

$$2x\cdot\frac{dx}{dt}+2y\cdot\frac{dy}{dt}=2D\cdot\frac{dD}{dt}$$

Now, solve for $\frac{dD}{dt}$:

$\frac{dx}{dt}=8$ is given, $\frac{dy}{dt}=0$ since y is constant. $y=100$ and $D=200$.

So, $x=\sqrt{(200)^2-(100)^2}=\sqrt{40000-10000}=\sqrt{30000}=100\sqrt{3}$.

Get $\frac{dD}{dt}=\frac{2(100\sqrt{3})(8)+2(100)(0)}{2(200)}=\frac{1600\sqrt{3}}{400}=\boxed{4\sqrt{3}\text{ ft/sec}}$

9. (20 pts) Answer the following for the function $f(x) = \frac{x^3}{3 - x^2}$ $\left(f'(x) = \frac{9x^2 - x^4}{(3 - x^2)^2} \right)$

(a) (2 pts) State any vertical asymptotes and show them on the graph below.

Vertical asymptotes when $3 - x^2 = 0 \Rightarrow x^2 = 3 \Rightarrow \boxed{x = \sqrt{3}}, \boxed{x = -\sqrt{3}}$

(b) (5 pts) Verify the intervals on which $f(x)$ is increasing and on which $f(x)$ is decreasing.

$f'(x) = 0$ when $9x^2 - x^4 = 0 \Rightarrow x^2(9 - x^2) = 0 \Rightarrow \boxed{x = 0}$ or $\boxed{x = 3}$ or $\boxed{x = -3}$

$f'(x)$ DNE when $(3 - x^2)^2 = 0 \Rightarrow \boxed{x = \sqrt{3}}$ or $\boxed{x = -\sqrt{3}}$

Sign Chart:

$f'(x)$	⊖	0	⊕	DNE	⊕	0	⊕	DNE	⊕	0	⊖
x	↑	-3	↑	$-\sqrt{3}$	↑	0	↑	$\sqrt{3}$	↑	3	↑
	-100		-2		-1		1		2		100

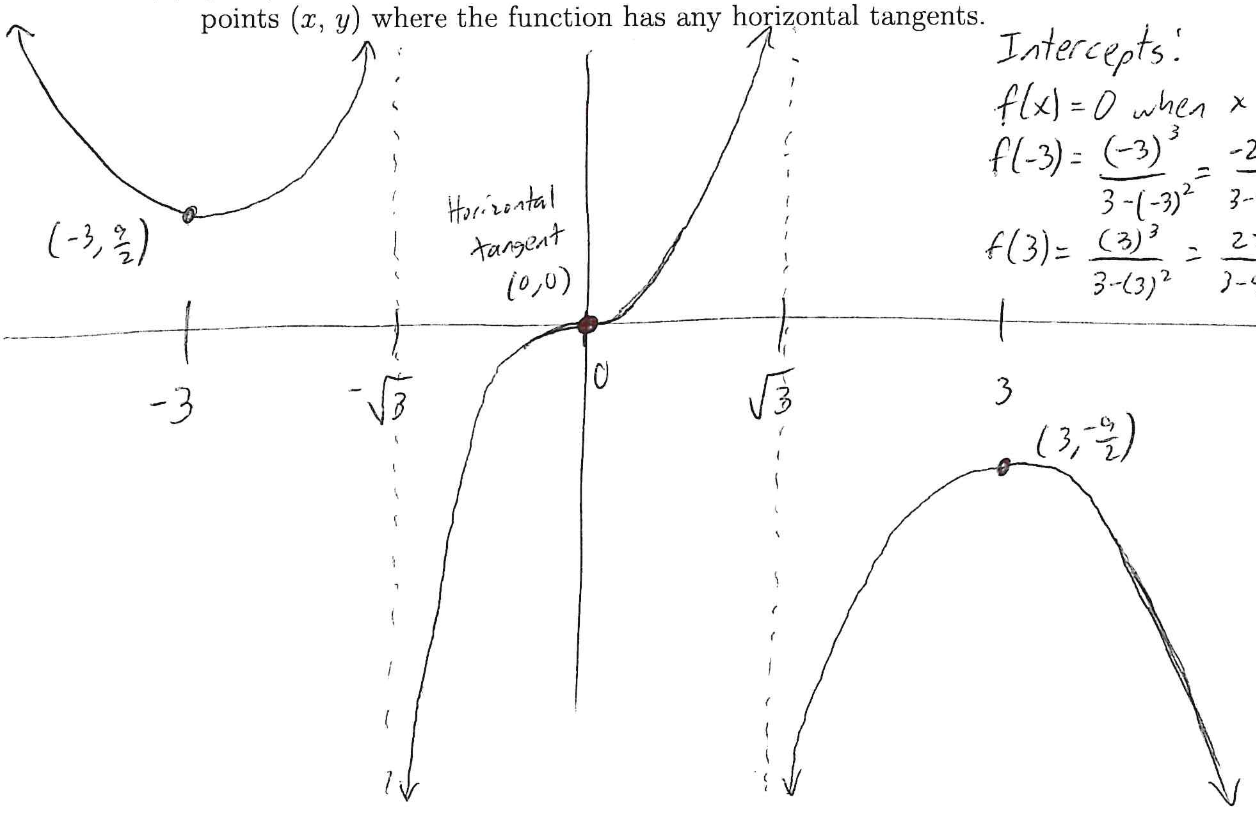
$f(x)$ is increasing on $(-3, -\sqrt{3}) \cup (-\sqrt{3}, 0) \cup (0, \sqrt{3}) \cup (\sqrt{3}, 3)$

$f(x)$ is decreasing on $(-\infty, -3) \cup (3, \infty)$

(c) (5 pts) $\lim_{x \to -\infty} f(x) =$ $-\infty$ $\qquad$ $\lim_{x \to \infty} f(x) =$ ∞

$= \lim_{x \to -\infty} \frac{x}{3/x^2 - 1} = -\infty$ $\qquad$ $= \lim_{x \to \infty} \frac{x}{3/x^2 - 1}$

(d) (8pts) Graph the function. Show asymptote(s), intercept(s) and the exact points (x, y) where the function has any horizontal tangents.



Intercepts:

$f(x) = 0$ when $x^3 = 0 \Rightarrow \boxed{x = 0}$

$f(-3) = \frac{(-3)^3}{3 - (-3)^2} = \frac{-27}{3 - 9} = \boxed{\frac{+9}{2}}$

$f(3) = \frac{(3)^3}{3 - (3)^2} = \frac{27}{3 - 9} = \boxed{\frac{-9}{2}}$

10. (15 pts) A farmer wants to build a rectangular shed with vertical walls and horizontal roof. The length of the shed must be twice the width. He needs the shed to be 288 cubic yards. Find the dimensions of the shed that uses the least possible amount of plywood for the walls and the roof. (The base is cement so it does not use any plywood.)

Objective: Minimize Area $= lw + 2lh + 2wh$

Constraint: $V = lwh = 288 \Rightarrow 2w^2h = 288 \Rightarrow h = \frac{144}{w^2}$

Constraint: $l = 2w$

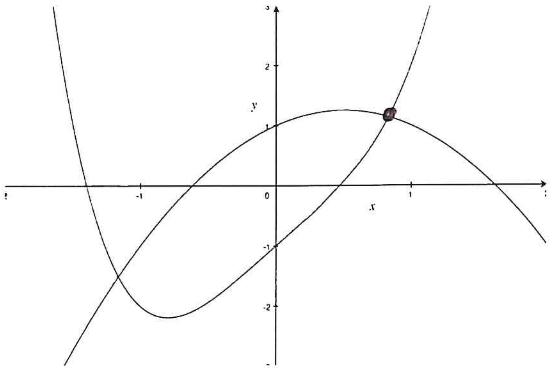
$$\text{So, } A(w) = 2w^2 + 4w \cdot \frac{144}{w^2} + 2w \cdot \frac{144}{w^2}$$

$$= 2w^2 + \frac{6 \cdot 144}{w}$$

$$A'(w) = 4w + \frac{-6 \cdot 144}{w^2} = 0 \Rightarrow 4w = \frac{6 \cdot 144}{w^2} \Rightarrow w^3 = \frac{6 \cdot 144}{4}$$

$$\Rightarrow \boxed{w = \sqrt[3]{\frac{6 \cdot 144}{4}}}, \quad \boxed{l = 2 \cdot \sqrt[3]{\frac{6 \cdot 144}{4}}}, \quad \boxed{h = \frac{144}{\left(\sqrt[3]{\frac{6 \cdot 144}{4}}\right)^2}}$$

11. (10 pts) Use Newton's Method once to approximate the first postitive value where the functions $f(x) = x^4 + 2x - 1$ and $g(x) = 1 + x - x^2$ intersect.



Formula: $x_1 = x_0 - \frac{h(x_0)}{h'(x_0)}$

Here, $f(x) = g(x)$ when $f(x) - g(x) = 0$

$\Rightarrow x^4 + 2x - 1 - 1 - x + x^2 = 0$

$\Rightarrow h(x) := x^4 + x^2 + x - 2 = 0$

Looks like $x=1$ is close to the intersection point, so guess $x_0 = 1$.

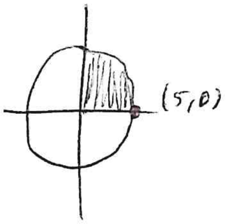
Then $h(x_0) = h(1) = (1)^4 + (1)^2 + (1) - 2 = \boxed{1}$

$h'(x) = 4x^3 + 2x + 1$, so $h'(x_0) = 4(1)^3 + 2(1) + 1 = \boxed{7}$.

Result: $x_1 = 1 - \frac{1}{7} = \boxed{\frac{6}{7}}$.

12. (10 pts) Evaluate the following integrals. You may use geometry.

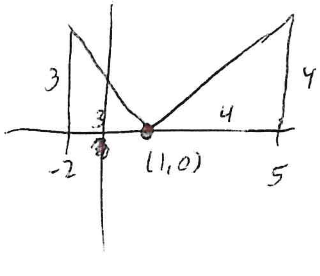
(a) $\int_0^5 \sqrt{25 - x^2}\, dx = A$



Since $y = \sqrt{25 - x^2} \Rightarrow y^2 = 25 - x^2 \Rightarrow x^2 + y^2 = 25$, this is a circle with radius 5. We take only the top half and rightside to get $A = \frac{1}{4}\pi(5)^2 = \boxed{\frac{25}{4}\pi}$

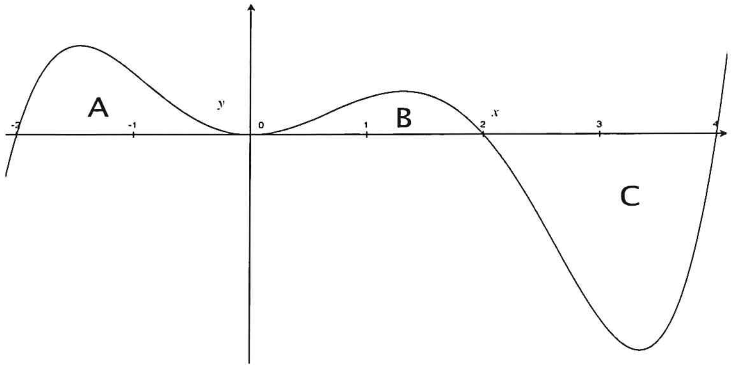
(positive squareroot) $0 \leq x \leq 5$

(b) $\int_{-2}^5 |x - 1|\, dx$

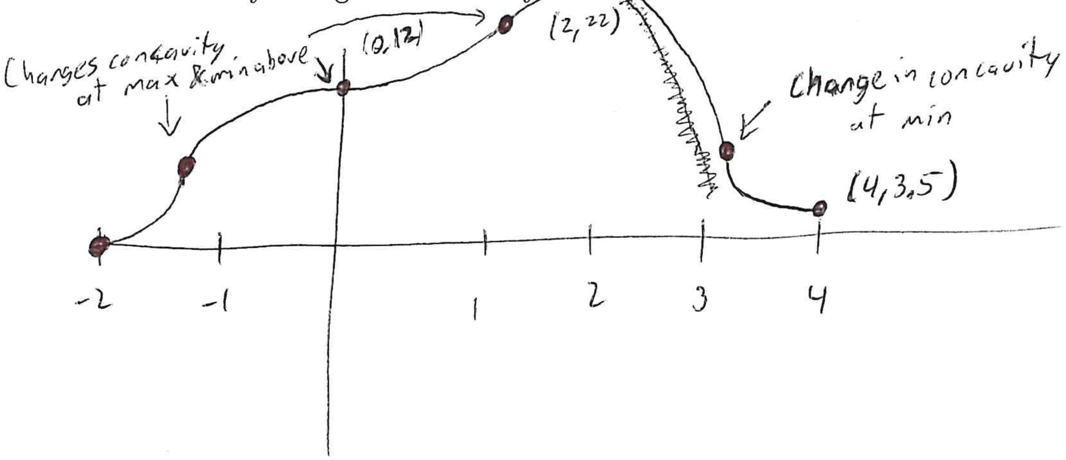


Area $= \frac{1}{2}(3)(3) + \frac{1}{2}(4)(4) = \frac{9}{2} + \frac{16}{2} = \boxed{\frac{25}{2}}$

13. (15 pts) Below is the plot of $f'(x)$. The area of section(A) is 12. The area of section(B) is 10. The area of section(C) is 18.5.



(a) (7 pts) Sketch a graph of $f(x)$ on $-2 \leq x \leq 4$ such that $f(-2) = 0$. Make sure to show where $f(x)$ has local maximum values, local mininimum values and any changes in concavity.



(b) (2 pts each) Determine the values below.

$f(2) =$ 22

$f(4) =$ 3.5

$\int_0^2 f'(x)\,dx =$ 10

$= f(2) - f(0)$

$= 22 - 12 = 10$

$\int_0^4 f'(x)\,dx =$ -8.5

$= f(4) - f(0)$

$= 3.5 - 12 = -8.5$

14. (30 pts)Evaluate the following integrals.

(a) $\int_1^4 \left(\frac{4}{x^2} + 3\sqrt{x} - \frac{3}{\sqrt{x}} + 2\right)\, dx$ (Simplify answer)

$$= \int_1^4 (4x^{-2} + 3x^{1/2} - 3x^{-1/2} + 2)\,dx = \left[\frac{4x^{-1}}{-1} + \frac{2\cdot 3x^{3/2}}{3} - \frac{2\cdot 3x^{1/2}}{1} + 2x\right]_1^4$$

$$= \left(\frac{-4}{(4)} + 2(4)^{3/2} - 6(4)^{1/2} + 2(4)\right) - \left(\frac{-4}{(1)} + 2(1)^{3/2} - 6(1)^{1/2} + 2(1)\right)$$

$$= -1 + 8 - 12 + 8 + 4 - 2 + 6 - 2 = \boxed{9}$$

(b) $\int \arcsin(x)\, dx$ ← Parts

$$\int f \cdot dg = f \cdot g - \int g \cdot df$$

$f = \arcsin(x)$ $\quad dg = 1 \cdot dx$

$df = \frac{1}{\sqrt{1-x^2}}\,dx$ $\quad g = x$

$$= x\arcsin(x) - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

$u = 1 - x^2$

$du = -2x\,dx$

$dx = \frac{du}{-2x}$

$$= x\arcsin(x) - \int \frac{x}{\sqrt{u}} \cdot \frac{du}{-2x} = x\arcsin(x) + \int \frac{1}{2} u^{-1/2}\,du$$

$$= x\arcsin(x) + \frac{1}{2}\frac{u^{1/2}}{1/2} + C = \boxed{x\arcsin(x) + \sqrt{1-x^2} + C}$$

(c) $\int \frac{(\ln(x))^3}{x}\, dx$ ← u-sub

$u = \ln(x)$

$du = \frac{1}{x}\,dx$

$$= \int u^3\,du = \frac{1}{4}u^4 + C = \boxed{\frac{1}{4}(\ln(x))^4 + C}$$

u-sub

(d) $\int 4\sin^3(2x)\cos^2(2x)\,dx$

w-sub

$$\boxed{\begin{array}{l} w = \cos(u) \\ dw = -\sin(u)\,du \end{array}}$$

$$\boxed{\begin{array}{l} u = 2x \\ du = 2dx \\ dx = \frac{du}{2} \end{array}}$$

$$= \int 4\sin^3(u)\cos^2(u)\frac{du}{2} = 2\int \sin^3(u)\cos^2(u)\,du$$

$$= 2\int -\sin^2(u)\,w^2\,dw = -2\int(1-\cos^2(u))\,w^2\,dw = -2\int(1-w^2)w^2\,dw$$

$$= -2\int(w^2-w^4)\,dw = -2\left(\frac{w^3}{3}-\frac{w^5}{5}\right)+C = -2\left(\frac{\cos^3(u)}{3}-\frac{\cos^5(u)}{5}\right)+C$$

$$\boxed{= -2\left(\frac{\cos^3(2x)}{3}-\frac{\cos^5(2x)}{5}\right)+C}$$

(e) $\int \frac{3x}{\sqrt{1-4x^2}}\,dx$

u-sub

$$\boxed{\begin{array}{l} u = 1-4x^2 \\ du = -8x\,dx \\ dx = \frac{du}{-8x} \end{array}}$$

$$= \int \frac{3x}{\sqrt{u}}\cdot\frac{du}{-8x} = \frac{-3}{8}\int u^{-1/2}\,du = \frac{-3}{8}\cdot\frac{u^{1/2}}{1/2}+C$$

$$\boxed{= \frac{-3}{4}(1-4x^2)^{1/2}+C}$$

Parts

(f) $\int 2x\,e^{-2x}\,dx$

$$\boxed{\int f\cdot dg = f\cdot g - \int g\cdot df}$$

$$\boxed{\begin{array}{ll} f = 2x & dg = e^{-2x}dx \\ df = 2dx & g = \frac{-1}{2}e^{-2x} \end{array}}$$

u-sub
u=2x

$$= -xe^{-2x} - \int -e^{-2x}\,dx$$

$$= -xe^{-2x} + \int e^{-2x}\,dx$$

$$\boxed{= -xe^{-2x} - \frac{1}{2}e^{-2x} + C}$$

u-sub
u=2x