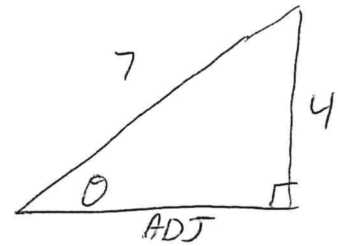
Solutions

1. (20 pts) Answer the following



(a) (3 pts) Determine the numerical value of $\tan\left(\arcsin\left(\frac{4}{7}\right)\right)$.

See picture. $\sin\theta = \frac{4}{7}$, we want $\tan\theta = \frac{OPP}{ADJ}$.

$ADJ = \sqrt{7^2-4^2} = \sqrt{49-16} = \sqrt{33}$, get $\tan\theta = \boxed{\frac{4}{\sqrt{33}}}$.

(b) (7 pts) A bacteria culture grows with a constant relative growth rate. After 2 hours there are 80 bacteria and after 5 hours the count is 5,120. Find an expression for the population after t hours.

Growth formula: $A = Pe^{rt}$. So, $80 = Pe^{2r}$ and $5120 = Pe^{5r}$.

Now, divide: $\frac{5120}{80} = \frac{Pe^{5r}}{Pe^{2r}} \Rightarrow 64 = e^{3r} \Rightarrow \boxed{r = \frac{\ln(64)}{3}}$. So, $80 = Pe^{2\frac{\ln(64)}{3}}$

$\Rightarrow P = \frac{e^{\left(\frac{2\ln(64)}{3}\right)}}{80}$.

Get $\boxed{A = \frac{e^{\left(\frac{2\ln(64)}{3}\right)}}{80} \cdot e^{\frac{\ln(64)}{3}t}}$

(c) (5 pts) Determine the inverse of $f(x) = \dfrac{3x}{2x+3}$.

The inverse is defined by $x = \frac{3y}{2y+3} \Rightarrow 2yx+3x = 3y \Rightarrow 2yx - 3y = -3x$

$\Rightarrow y(2x-3) = -3x \Rightarrow \boxed{y = \frac{-3x}{+2x-3}}$.

(d) (5 pts) If $f(x) = x^3 + 4x - 5$, then $f(1) = 0$. Determine $(f^{-1})'(0)$.

Inverse function theorem: $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$. Note $f'(x) = 3x^2+4$.

So, $(f^{-1})'(0) = \frac{1}{f'(f^{-1}(0))} = \frac{1}{f'(1)} = \frac{1}{3(1)^2+4} = \boxed{\frac{1}{7}}$

(note $f^{-1}(0) = 1$)

2. (15 pts) For the curve $e^{2x+y} + x^3 + y^2 = 8$,

(a) (10 pts) determine the linearization (equation of the tangent line) to the function at the point $(1, -2)$.

Formula: $f(x) \approx f(a) + f'(a)(x-a)$. Since we don't have a function, "$f(a)$" $= -2$ (the y-coordinate we're given) and "$f'(a)$" is $\frac{dy}{dx}$ evaluated at the given point $(1,-2)$. So, compute by differentiating the curve with respect to x:

$e^{2x+y} \cdot \left(2 + \frac{dy}{dx}\right) + 3x^2 + 2y\frac{dy}{dx} = 0 \Rightarrow 2e^{2x+y} + \frac{dy}{dx} \cdot e^{2x+y} + 3x^2 + 2y\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx}\left(e^{2x+y} + 2y\right) = -3x^2 - 2e^{2x+y} \Rightarrow \boxed{\frac{dy}{dx} = \frac{-3x^2 - 2e^{2x+y}}{e^{2x+y} + 2y}}$. At $(1,-2)$, get $\frac{dy}{dx} = \frac{-3(1)^2 - 2e^0}{e^0 + 2(-2)}$

$= \frac{-3-2}{1-4} = \frac{-5}{-3} = \boxed{\frac{5}{3}}$

$\boxed{\text{Conclude: } \boxed{f(x) \approx -2 + \frac{5}{3}(x-1)}}$

(b) (5 pts) Use the linearization to approximate y when $x = 1.3$.

$f(1.3) \approx -2 + \frac{5}{3}(1.3-1) = -2 + \frac{5}{3} \cdot \frac{3}{10} = -2 + \frac{1}{2} = \boxed{-\frac{3}{2}}$

3. (10 pts) Use Newton's Method once to approximate $\sqrt[3]{60}$. $= x \Rightarrow x^3 = 60 \Rightarrow x^3 - 60 = 0$.

Since $\sqrt[3]{64} = 4$, let's solve $f(x) = x^3 - 60 = 0$ with $x_0 = 4$ as our first guess.

Formula: $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$. Get $f(x_0) = (4)^3 - 60 = 64 - 60 = 4$,

$f'(x) = 3x^2$, so $f'(x_0) = 3(4)^2 = 48$.

Result: $x_1 = 4 - \frac{4}{48} = 4 - \frac{1}{12} = \boxed{\frac{47}{12}}$

4. (5 pts each=40 pts) Differentiate the following. Do not simplify your answer.

(a) $f(x) = \sin^5(3x)$ ✓ Chain

Note $\sin^5(3x) = [\sin(3x)]^5$. Get $\boxed{f'(x) = 5\sin^4(3x) \cdot \cos(3x) \cdot 3}$

(b) $y = \dfrac{3e^{2x}}{x^2+1}$ ✓ Quotient

$$\boxed{y' = \frac{[x^2+1] \cdot [3e^{2x} \cdot 2] - [3e^{2x}] \cdot [2x]}{(x^2+1)^2}}$$

(c) $f(x) = \ln(4x + 3\cos(2x))$

$$\boxed{f'(x) = \frac{1}{4x+3\cos(2x)} \cdot (4 + 3(-\sin(2x) \cdot 2))}$$

(d) $y = \arctan(2^x)$

$$\boxed{y' = \frac{1}{1+(2^x)^2} \cdot (2^x \ln(2))}$$

(e) $f(x) = \sqrt[3]{x - 3x^2 + 4x^3}$ $= (x - 3x^2 + 4x^3)^{1/3}$

$$\boxed{f'(x) = \frac{1}{3}(x - 3x^2 + 4x^3)^{-2/3} \cdot (1 - 6x + 12x^2)}$$

↓ Product!

(f) $y = (7x + 3)^4(x^2 + 1)^3$

$$\boxed{y' = \left\{\left[(7x+3)^4\right] \cdot \left[3(x^2+1)^2 \cdot 2x\right]\right\} + \left\{\left[4(7x+3)^3 \cdot 7\right] \cdot \left[(x^2+1)^3\right]\right\}}$$

(g) $f(x) = \dfrac{1000}{(1000 + 10e^{-0.1x})^2}$ $= 1000(1000 + 10e^{-0.1x})^{-2}$

$$\boxed{f'(x) = -2000(1000 + 10e^{-0.1x})^{-3} \cdot \left[10e^{-0.1x} \cdot (-0.1)\right]}$$

(h) $y = \displaystyle\int_{\pi/4}^{x^2} \cos^4(6u)\sin^4(6u)\,du$

$$\boxed{y' = \cos^4(6x^2)\sin^4(6x^2) \cdot 2x - \cos^4\left(6 \cdot \frac{\pi}{4}\right)\sin^4\left(6 \cdot \frac{\pi}{4}\right) \cdot 0}$$

5. (20 pts) Determine the limit (show all work).

(a) $\lim_{x\to 1} \frac{\ln(1+4\ln(x))}{x^2-1}$ $\to \frac{\ln(1)=0}{1-1=0}$ INDETERMINATE

$$\overset{L'H}{\downarrow} = \lim_{x\to 1} \frac{\frac{1}{1+4\ln(x)} \cdot \frac{4}{x}}{2x} = \frac{\frac{1}{1}\cdot\frac{4}{1}}{2\cdot 1} = \frac{4}{2} = \boxed{2}$$

(b) $\lim_{x\to 0} (1+\arcsin(2x))^{1/x}$ $\to 1^\infty$ INDETERMINATE

$$L = \lim_{x\to 0} (1+\arcsin(2x))^{1/x} \Rightarrow \ln(L) = \lim_{x\to 0} \ln\left((1+\arcsin(2x))^{1/x}\right) = \lim_{x\to 0} \frac{\ln(1+\arcsin(2x))}{x}$$

$$\overset{L'H}{\downarrow} = \lim_{x\to 0} \frac{\frac{1}{1+\arcsin(2x)} \cdot \frac{1}{\sqrt{1-(2x)^2}} \cdot 2}{1} = \frac{\frac{1}{1}\cdot\frac{1}{\sqrt{1}}\cdot 2}{1} = 2. \text{ Thus, } \boxed{L = e^2}.$$

(c) $\lim_{x\to 3^-} \frac{|x^2-5x+6|}{x-3}$

$$= \lim_{x\to 3^-} \frac{|(x-3)(x-2)|}{x-3}.$$ When $x<3$, (and x is close to 3) $(x-3)(x-2)<0$. Thus get

$$= \lim_{x\to 3^-} \frac{-(x-3)(x-2)}{x-3} = \lim_{x\to 3^-} -(x-2) = -(3-2) = \boxed{-1}.$$

(d) $\lim_{x\to\infty} \left[\ln(4x^2-3x+1) - \ln(x^2+7x+80)\right]$

$$= \lim_{x\to\infty} \ln\left(\frac{4x^2-3x+1}{x^2+7x+80}\right) = \ln\left(\lim_{x\to\infty} \frac{4x^2-3x+1}{x^2+7x+80}\right)$$

$$= \ln\left(\lim_{x\to\infty}\left(\frac{4-\frac{3}{x}+\frac{1}{x^2}}{1+\frac{7}{x}+\frac{80}{x^2}}\right)\right) = \boxed{\ln(4)}$$

6. (10 pts) 40π in^3 of dough is being rolled so that it remains cylindrical. The length is increasing at 1/2 inches per second. At what rate is the radius changing when the length is 10 inches?

As the dough is rolled, volume is constant. So,

$V = 40\pi = \pi r^2 l \Rightarrow 40 = r^2 l$. Now, differentiate w.r.t. time

$0 = \left\{2r \left[\frac{dr}{dt}\right]\right\} \cdot \{l\} + \{r^2\} \cdot \left[\frac{dl}{dt}\right].$

When $l = 10$, $40 = r^2(10) \Rightarrow r^2 = 4 \Rightarrow \boxed{r = +2}$. $\frac{dl}{dt} = \frac{1}{2}$ is given, so get

$$\frac{dr}{dt} = \frac{-(2)^2 \cdot \frac{1}{2}}{2(2)(10)} = \frac{-2}{40} = \boxed{\frac{-1}{20} \text{ in./sec}}.$$

7. (10 pts) Determine the range (minimum to maximum values) of the function $f(x) = x^2\sqrt{100 - x^2}$ on the interval $[5, 10]$.

To find the range, we find the max & min values on the interval.

So, find where $f'(x) = 0$ or DNE (critical points) in the interval, and test these and the endpoints of the interval ($x = 5$ and $x = 10$) in $f(x)$.

Note $f'(x) = 2x\sqrt{100 - x^2} + x^2 \cdot \frac{-2x}{2\sqrt{100 - x^2}} = 2x\sqrt{100 - x^2} - \frac{x^3}{\sqrt{100 - x^2}}.$

$f'(x)$ DNE when $\sqrt{100 - x^2} = 0 \Rightarrow \cancel{x = -10}$ or $\boxed{x = 10}$.

Not in interval.

$f'(x) = 0$ when $2x\sqrt{100 - x^2} = \frac{x^3}{\sqrt{100 - x^2}} \Rightarrow 2x(100 - x^2) = x^3 \Rightarrow 200x - 2x^3 = x^3$

$\Rightarrow 200x - 3x^3 = 0 \Rightarrow x(200 - 3x^2) = 0 \Rightarrow \cancel{x = 0}$ or $x = \boxed{\sqrt{\frac{200}{3}}}$ or $\cancel{x = -\sqrt{\frac{200}{3}}}$

Not in interval   Not in interval

Get $f(5) = (5)^2\sqrt{100 - (5)^2} = 25\sqrt{75} = 25 \cdot 5\sqrt{3} = \boxed{125\sqrt{3}}$

$f(10) = (10)^2\sqrt{100 - (10)^2} = \boxed{0}$

$f\left(\sqrt{\frac{200}{3}}\right) = \frac{200}{3}\sqrt{100 - \frac{200}{3}} = \frac{200}{3}\sqrt{\frac{100}{3}} = \frac{2000}{3\sqrt{3}} = \boxed{\frac{2000}{9}\sqrt{3}}.$

$\boxed{\text{Range is } 0 < f(x) < \frac{2000}{9}\sqrt{3} \text{ on } [5, 10]}$

8. (20pts) Given the function $f(x) = 8x^2e^{-x}$.

(a) (5 pts) State the exact points (x and y coordinates) where the function has any local maximum or local minimum values.

$f'(x) = -8x^2e^{-x} + 16xe^{-x} = 0 \Rightarrow 8xe^{-x}(-x+2) = 0 \Rightarrow 8x = 0$ or $e^{-x} = 0$ or $-x+2=0$

$\boxed{x=0}$ No solutions $\boxed{x=2}$

Sign Chart: $f'(x)$ ⊖ 0 ⊕ 0 ⊖

x: -1, 0, 1, 2, 3

$\boxed{\begin{array}{l}\text{Local max at } x=2,\ y = 8(2)^2e^{-2} = \frac{32}{e^2} \\ \text{Local min at } x=0,\ y=8(0)^2e^{-0} = 0\end{array}}$

(b) (5 pts) State the values of x where $f(x)$ changes concavity

$f''(x) = 8x^2e^{-x} - 16xe^{-x} + 16e^{-x} - 16xe^{-x} = (8x^2-32x+16)e^{-x} = 0 \Rightarrow 8x^2-32x+16=0$

$\Rightarrow x^2-4x+2=0 \Rightarrow x = \frac{4 \pm \sqrt{16-4(1)(2)}}{2} = 2 \pm \frac{\sqrt{8}}{2} = \boxed{2 \pm \sqrt{2}}$

$\boxed{f(x) \text{ changes concavity at } \boxed{x = 2+\sqrt{2}} \text{ and } \boxed{x = 2-\sqrt{2}}}$

Sign Chart: $f''(x)$ ⊕ 0 ⊖ 0 ⊕

x: -100, $2-\sqrt{2}$, 1, $2+\sqrt{2}$, 100

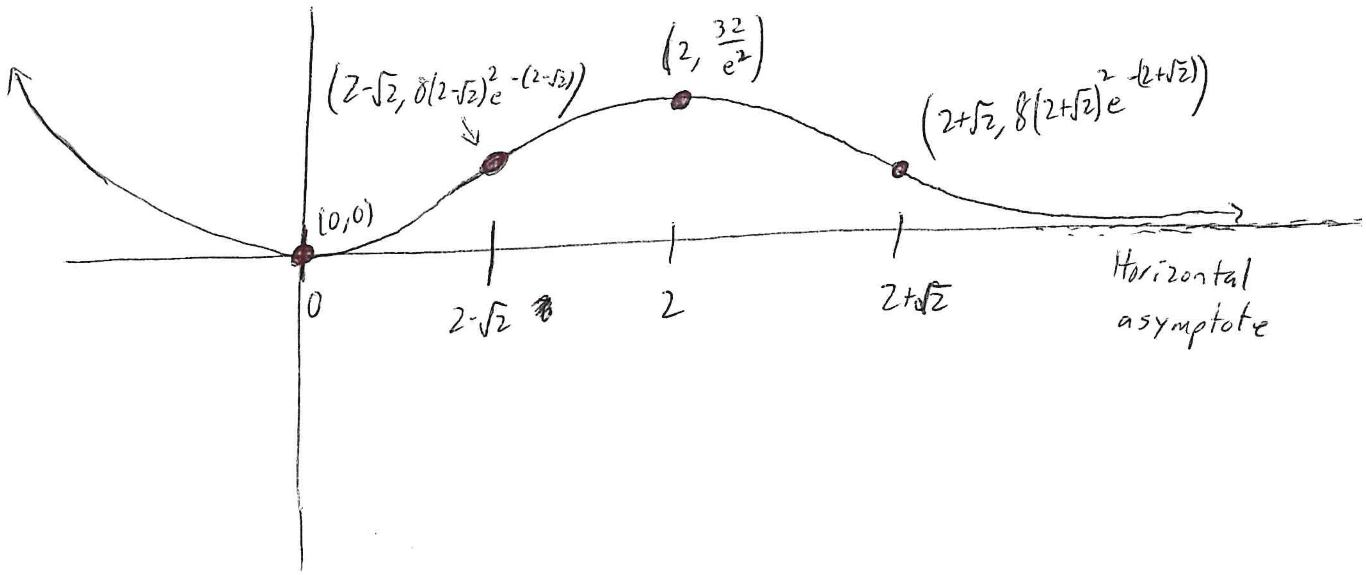
(c) (4 pts) $\lim_{x\to-\infty} f(x) =$ ∞ $\qquad \lim_{x\to\infty} f(x) =$ 0

(d) (6 pts) Graph the function $f(x) = 8x^2e^{-x}$ showing parts (a)-(c).

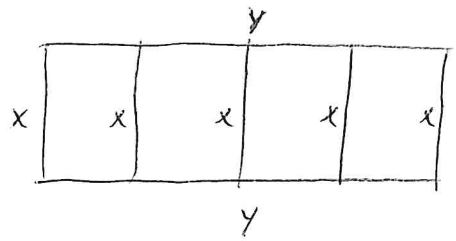
(c) $\lim_{x\to-\infty} 8x^2e^{-x} \to 8(-\infty)^2e^{-(-\infty)} \to \infty \cdot e^{\infty} \to \infty \cdot \infty \to \boxed{\infty}$

$\lim_{x\to\infty} 8x^2e^{-x} = \lim_{x\to\infty} \frac{8x^2}{e^x} \to \frac{\infty}{\infty} \overset{L'H}{=} \lim_{x\to\infty} \frac{16x}{e^x} \overset{L'H}{=} \lim_{x\to\infty} \frac{16}{e^x} \to \frac{16}{\infty} = \boxed{0}$

(d)



9. (15 pts) A farmer has a total of 750 ft of fence. He wants to enclose a rectangular area and then divide it into four parts by running fence parallel to one side of the rectangle. What is the largest possible total area of the four pens?



Objective: Maximize $A = xy$

Constraint: $2y + 5x = 750 \Rightarrow x = \frac{750-2y}{5}$

So, $A(y) = \frac{750-2y}{5} \cdot y = 150y - \frac{2}{5}y^2$.

$A'(y) = 150 - \frac{4}{5}y = 0 \Rightarrow \frac{4}{5}y = 150 \Rightarrow y = \frac{750}{4} = \boxed{\frac{375}{2}}$

$x = \frac{750 - 2 \cdot \frac{375}{2}}{5} = \frac{750-375}{5} = \boxed{\frac{375}{5}}$

Thus, Area $= \frac{375}{2} \cdot \frac{375}{5} = \boxed{\frac{(375)^2}{10}}\ \boxed{ft^2}$

10. (5 pts each=40 pts) Integrate the following.

(a) $\displaystyle\int_3^4 \frac{x}{\sqrt{25-x^2}}\,dx$ u-sub

$u = 25 - x^2$

$du = -2x\,dx$

$dx = \dfrac{du}{-2x}$

lower $= \sqrt{25-(3)^2} = 4$ upper $= \sqrt{25-(4)^2} = 3$

$$= \int_4^3 \frac{x}{\sqrt{u}} \cdot \frac{du}{-2x} = -\frac{1}{2}\int_4^3 u^{-1/2}\,du = \left[-\frac{1}{2}\,\frac{u^{1/2}}{1/2}\right]_4^3$$

$$\boxed{= \left(-(3)^{1/2}\right) - \left(-(4)^{1/2}\right)}$$

(b) $\displaystyle\int 3x^4 \ln(x)\,dx$ Parts

$$\boxed{\int f \cdot dg = f \cdot g - \int g \cdot df}$$

$f = \ln(x)$ $dg = 3x^4\,dx$

$df = \frac{1}{x}\,dx$ $g = \frac{3}{5}x^5$

$$= \frac{3}{5}x^5\ln(x) - \int \frac{3}{5}x^5 \cdot \frac{1}{x}\,dx = \frac{3}{5}x^5\ln(x) - \frac{3}{5}\int x^4\,dx$$

$$\boxed{= \frac{3}{5}x^5\ln(x) - \frac{3}{25}x^5 + C}$$

(c) $\displaystyle\int \cos^3(x)\,dx$ u-sub

$u = \sin(x)$

$du = \cos(x)\,dx$

$$= \int \cos^2(x)\,du = \int (1 - \sin^2(x))\,du = \int (1-u^2)\,du = u - \frac{u^3}{3} + C$$

$$\boxed{= \sin(x) - \frac{\sin^3(x)}{3} + C}$$

(d) $\displaystyle\int \frac{2x+3}{2x^2+6x+5}\,dx$ u-sub

$u = 2x^2 + 6x + 5$

$du = (4x+6)\,dx$

$dx = \dfrac{du}{4x+6}$

$$= \int \frac{2x+3}{u} \cdot \frac{du}{4x+6} = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C$$

$$\boxed{= \frac{1}{2}\ln|2x^2+6x+5| + C}$$

(e) $\int \arcsin(x)\,dx$ Parts

$$\boxed{\int f\cdot dg = f\cdot g - \int g\cdot df}$$

$$\boxed{\begin{array}{ll} f=\arcsin(x) & dg = 1\,dx \\ df = \frac{1}{\sqrt{1-x^2}}\,dx & g = x \end{array}} = x\arcsin(x) - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

u-sub

$$\boxed{\begin{array}{l} u = 1-x^2 \\ du = -2x\,dx \\ dx = \frac{du}{-2x} \end{array}}$$

$$= x\arcsin(x) - \int \frac{x}{\sqrt{u}}\cdot\frac{du}{-2x} = x\arcsin(x) + \frac{1}{2}\int u^{-1/2}\,du$$

$$= x\arcsin(x) + \frac{1}{2}\cdot\frac{u^{1/2}}{1/2} + C = \boxed{x\arcsin(x) + \sqrt{1-x^2} + C}$$

Double-angle

(f) $\int \sin^2(3x)\,dx$

Formula: $\sin^2(\theta) = \frac{1}{2}(1-\cos(2\theta))$.

Get $\int \sin^2(3x)\,dx = \frac{1}{2}\int (1-\cos(6x))\,dx = \boxed{\frac{1}{2}x - \frac{1}{2}\cdot\frac{1}{6}\sin(6x) + C}$

u-sub
$u=6x$

(g) $\int \sec^3(x)\tan(x)\,dx$

$$\boxed{\begin{array}{l} u=\sec(x) \\ du = \sec(x)\tan(x)\,dx \end{array}} = \int u^2\,du = \frac{u^3}{3} + C = \boxed{\frac{\sec^3(x)}{3} + C}$$

(h) $\int \frac{3}{\sqrt{1-4x^2}}\,dx$ Looks like the derivative of arcsin(x)...

$$\boxed{\begin{array}{l} u=2x \\ du = 2\,dx \\ dx = \frac{du}{2} \end{array}} = \int \frac{3}{\sqrt{1-u^2}}\cdot\frac{du}{2} = \frac{3}{2}\int\frac{1}{\sqrt{1-u^2}}\,du = \frac{3}{2}\arcsin(u) + C$$

$$\boxed{= \frac{3}{2}\arcsin(2x) + C}$$