Solutions

1. (18 points) Evaluate the following limits.

(a) $\lim_{x\to 2^-} \frac{|x^2+x-6|}{x-2}$

$$= \lim_{x\to 2^-} \frac{|(x-2)(x+3)|}{x-2}$$

When $x<2$, $(x-2)(x+3)<0$, so flip sign:

$$= \lim_{x\to 2^-} \frac{-(x-2)(x+3)}{x-2}$$

$$= \lim_{x\to 2^-} -(x+3) = -(2+3) = \boxed{-5}$$

(b) $\lim_{x\to+\infty} 5x^2e^{-2x} \to \infty \cdot 0$ INDETERMINATE

$$= \lim_{x\to+\infty} \frac{5x^2}{e^{2x}}$$

$$\overset{L'H}{\downarrow}{=} \lim_{x\to\infty} \frac{10x}{2e^{2x}} = \lim_{x\to\infty} \frac{5x}{e^{2x}} \overset{L'H}{\downarrow}{=} \lim_{x\to\infty} \frac{5}{2e^{2x}} \Longrightarrow \frac{5}{2\cdot\infty} = \boxed{0}$$

(c) $\lim_{x\to+\infty} \left(1-\frac{1}{2x}\right)^x \to 1^\infty$ INDETERMINATE

Say $L = \lim_{x\to\infty} \left(1-\frac{1}{2x}\right)^x$. Then $\ln(L) = \lim_{x\to\infty}\left(\ln\left(\left(1-\frac{1}{2x}\right)^x\right)\right) = \lim_{x\to\infty} x \cdot \ln\left(1-\frac{1}{2x}\right)$

$$= \lim_{x\to\infty} \frac{\ln\left(1-\frac{1}{2x}\right)}{\frac{1}{x}} \overset{L'H}{\downarrow}{=} \lim_{x\to\infty} \frac{\frac{1}{1-\frac{1}{2x}} \cdot \frac{1}{(2x)^2} \cdot 2}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{\frac{1}{1-\frac{1}{2x}} \cdot \frac{1}{2} \cdot \frac{1}{x^2}}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{1}{1-\frac{1}{2x}} \cdot \frac{1}{2} = \boxed{\frac{1}{2}}$$

Conclude $\boxed{L = e^{1/2}}$

2. (36 points) Find the derivatives of the following functions. **You do NOT need to simplify your answer.**

(a) $f(x) = (x^2+1)^2 e^{3x}$

Product!

$$f'(x) = \left[(x^2+1)^2\right] \cdot \left[e^{3x} \cdot 3\right] + \left[2(x^2+1) \cdot 2x\right] \cdot \left[e^{3x}\right]$$

(b) $f(x) = \dfrac{\cos(3x)}{2x^2+1}$ Quotient

$$\boxed{f'(x) = \frac{\left[2x^2+1\right]\left[\sin(3x) \cdot 3\right] - \left[\cos(3x)\right] \cdot \left[4x\right]}{(2x^2+1)^2}}$$

(c) $f(x) = \tan(4x) + \tan^{-1}(4x)$

$$\boxed{f'(x) = \sec^2(4x) \cdot 4 + \frac{1}{1+(4x)^2} \cdot 4}$$

(d) $f(x) = (x^2 + 2x + 5)^{3x}$ ← Super-exponential

$$\ln(f(x)) = \ln\left((x^2+2x+5)^{3x}\right) = 3x \cdot \ln(x^2+2x+5)$$

$$\frac{1}{f(x)} \cdot f'(x) = \left[3x\right] \cdot \left[\frac{1}{x^2+2x+5} \cdot (2x+2)\right] + \left[3\right] \cdot \left[\ln(x^2+2x+5)\right]$$

$$\boxed{f'(x) = \left[3x \cdot \frac{2x+2}{x^2+2x+5} + 3\ln(x^2+2x+5)\right]\left((x^2+2x+5)^{3x}\right)}$$

(e) $f(x) = \int_{\sin x}^{1} \sqrt{1+t^2}\ dt$

$$\boxed{f'(x) = \sqrt{1+(1)^2} \cdot 0 - \sqrt{1+(\sin(x))^2} \cdot \cos(x)}$$

(f) Suppose that f is differentiable and

$$f(0) = 1, \quad f'(0) = 2, \quad f(1) = 2, \quad f'(1) = 3.$$

Let $g(x) = xf(x^2 - 1)$. Use this information to calculate $g(1)$ and $g'(1)$.

$$g(1) = 1 \cdot f((1)^2 - 1) = f(0) = \boxed{1}$$

$$g'(x) = [x] \cdot [f'(x^2-1) \cdot 2x] + [1] \cdot [f(x^2-1)]$$

$$g'(1) = 1 \cdot f'((1)^2-1) \cdot 2(1) + f((1)^2-1)$$

$$= 2 \cdot f'(0) + f(0) = 2 \cdot 2 + 1 = \boxed{5}$$

3. (36 points) Evaluate the following integrals.

(a) $\int \frac{3}{x^2+1} + \frac{x^2+1}{x} \, dx$

$$= \int 3 \cdot \frac{1}{x^2+1} \, dx + \int \frac{x^2}{x} \, dx + \int \frac{1}{x} \, dx$$

$$= 3\arctan(x) + \int x \, dx + \ln|x|$$

$$\boxed{= 3\arctan(x) + \frac{1}{2}x^2 + \ln|x| + C}$$

(b) $\int 3x^2 \ln x \, dx$ ↙ Parts

$$\boxed{\int f \cdot dg = f \cdot g - \int g \cdot df}$$

$$\boxed{\begin{matrix} f = \ln(x) & dg = 3x^2 dx \\ df = \frac{1}{x} dx & g = x^3 \end{matrix}}$$

$$= x^3 \ln(x) - \int x^3 \cdot \frac{1}{x} \, dx$$

$$= x^3 \ln(x) - \int x^2 dx$$

$$\boxed{= x^3 \ln(x) - \frac{1}{3} x^3 + C}$$

(c) $\int_0^1 \frac{4x}{x^2+1} \, dx$ ↙ u-sub

$$\boxed{\begin{matrix} u = x^2 + 1 \\ du = 2x \, dx \\ dx = \frac{du}{2x} \end{matrix}}$$

$$\boxed{\begin{matrix} \text{lower} = (0)^2 + 1 = 1 \\ \text{upper} = (1)^2 + 1 = 2 \end{matrix}}$$

$$= \int_1^2 \frac{4x}{u} \cdot \frac{du}{2x} = \int_1^2 \frac{2}{u} \, du = \Big[2\ln|u|\Big]_1^2 = 2\ln(2) - 2\ln(1)$$

$$\boxed{= 2\ln(2)}$$

(d) $\int \sin^2 x \cos^3 x \; dx$

$$\boxed{\begin{array}{l} u = \sin(x) \\ du = \cos(x)\,dx \end{array}}$$

$$= \int u^2 \cos^2(x)\,du = \int u^2 (1-\sin^2(x))\,du = \int u^2(1-u^2)\,du$$

$$= \int (u^2 - u^4)\,du = \frac{u^3}{3} - \frac{u^5}{5} + C = \boxed{\frac{\sin^3(x)}{3} - \frac{\sin^5(x)}{5} + C}$$

↙ Half-angle identity

(e) $\int 2\sin^2 x \; dx$

$$\boxed{\sin^2(\theta) = \frac{1}{2}(1-\cos(2\theta))}$$

$$= \int (1-\cos(2x))\,dx$$

$$\boxed{= x - \frac{1}{2}\sin(2x) + C}$$

↑

u-sub

$u = 2x$

↙ Parts

(f) $\int 6xe^{-x} \; dx$

$$\boxed{\int f \cdot dg = f \cdot g - \int g \cdot df}$$

$$\boxed{\begin{array}{ll} f = 6x & dg = e^{-x}\,dx \\ df = 6dx & g = -e^{-x} \\ & \uparrow \\ & \text{u-sub} \\ & u = -x \end{array}}$$

$$= -6xe^{-x} - \int -6e^{-x}\,dx$$

$$= -6xe^{-x} + 6\int e^{-x}\,dx$$

$$\boxed{= -6xe^{-x} - 6e^{-x} + C}$$

↑

u-sub

$u = -x$

4. (8 points) Find the values of a and b so that the following function is continuous on the real line $(-\infty, \infty)$.

$$f(x) = \begin{cases} x+1 & \text{if } x \le -1 \\ ax+b & \text{if } -1 < x < 1 \\ 4x^2 & \text{if } 1 \le x. \end{cases}$$

For $x \in (-\infty,-1) \cup (-1,1) \cup (1,\infty)$, $f(x)$ is continuous because it is a polynomial (or line). So, establish continuity at $x=-1$ and $x=1$ by showing $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x) = f(-1)$ and $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$:

$$\lim_{x\to -1^-} x+1 = \lim_{x\to -1^+} ax+b = -1+1 \quad \text{and} \quad \lim_{x\to 1^-} ax+b = \lim_{x\to 1^+} 4x^2 = 4(1)^2$$

$0 = -a+b$ and $a+b=4$

$\Rightarrow b=a$ $\Rightarrow$ $b+b=4 \Rightarrow \boxed{b=2}$

$\Rightarrow \boxed{a=2}$

5. (8 points) Use **the definition of derivative** (by means of the limit of difference quotient) to find the derivative of $f(x) = \sqrt{x+1}$. NO CREDIT will be given if the limit definition is not used.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{(x+h)+1} - \sqrt{x+1}}{h} \cdot \frac{\sqrt{(x+h)+1} + \sqrt{x+1}}{\sqrt{(x+h)+1} + \sqrt{x+1}}$$

$$= \lim_{h\to 0} \frac{(x+h+1)-(x+1)}{h\left(\sqrt{x+h+1} + \sqrt{x+1}\right)} = \lim_{h\to 0} \frac{h}{h\left(\sqrt{x+h+1} + \sqrt{x+1}\right)} = \lim_{h\to 0} \frac{1}{\sqrt{x+h+1} + \sqrt{x+1}}$$

$$\boxed{= \frac{1}{2\sqrt{x+1}}}$$

6. (10 points) At the point $(1, 0)$, determine the equation of the tangent line to the curve

$$x^2 - 3xy + y^2 = 1.$$

Differentiate with respect to x:

$$2x - 3\left(x \cdot \frac{dy}{dx} + 1 \cdot y\right) + 2y\frac{dy}{dx} = 0$$

$$2x - 3x\frac{dy}{dx} + 3y + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(2y - 3x) = -2x - 3y$$

$$\frac{dy}{dx} = \frac{-2x-3y}{2y-3x}$$

At (1,0), get $\frac{dy}{dx} = \frac{-2(1)-3(0)}{2(0)-3(1)} = \frac{-2}{-3}$

$= \boxed{\frac{2}{3}} \leftarrow$ Slope

The equation of the tangent line is $y - y_1 = m(x - x_1)$.

Since $x_1 = 1$, $y_1 = 0$, $m = \frac{2}{3}$, get

$$\boxed{y - 0 = \frac{2}{3}(x-1)}$$

7. (8 points) Let $f(x) = 3 + 2x^2 \ln x$.

(a) Determine the linearization of $f(x)$ at $x = 1$.

(b) Use the linearization to approximate $f(1.1)$.

Formula: $f(x) \approx f(a) + f'(a)(x-a)$, where $a=1$.

$f(1) = 3 + 2(1)^2 \ln(1) = 3 + 2 \cdot 0 = \boxed{3}$

$f'(x) = 4x \cdot \ln(x) + 2x^2 \cdot \frac{1}{x}$, so $f'(1) = 4\ln(1) + 2 = \boxed{2}$

So, $\boxed{f(x) \approx 3 + 2(x-1)}$, and $f(1.1) \approx 3 + 2(1.1-1) = 3 + 2(0.1) = \boxed{3.2}$

8. (8 points) Use Newton's Method ONCE starting with $x_1 = 0$ to approximate the solution to $e^{-x} - 3 = x$.

Formula: $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$. $f(x) = e^{-x} - 3 - x = 0$, so $f(0) = e^{-(0)} - 3 - 0$

$= 1 - 3 = \boxed{-2}$

$f'(x) = -e^{-x} - 1$, so $f'(0) = -e^{-(0)} - 1 = -1 - 1 = \boxed{-2}$

Get $x_2 = 0 - \frac{-2}{-2} = 0 - 1 = \boxed{-1}$

9. (10 points) Find the absolute maximum and minimum values of the function $f(x) = x\sqrt{4-x^2}$ on the interval $[-1, 2]$.

Step 1: Identify where $f'(x) = 0$ or DNE (critical points) in the interval.

Step 2: Plug in all critical points & interval endpoints into $f(x)$.

$f'(x) = 1 \cdot \sqrt{4-x^2} + x \cdot \frac{-2x}{2\sqrt{4-x^2}} = \sqrt{4-x^2}$ [illegible] $\frac{x^2}{\sqrt{4-x^2}}$. Note $f'(x)$ DNE when

$\sqrt{4-x^2} = 0 \Rightarrow 4 - x^2 = 0 \Rightarrow (2-x)(2+x) = 0 \Rightarrow \boxed{x=2}$ or $\cancel{\boxed{x=-2}}$.

↑ Not in interval.

$f'(x) = 0$ when $\sqrt{4-x^2} - \frac{x^2}{\sqrt{4-x^2}} = 0 \Rightarrow \sqrt{4-x^2} = \frac{x^2}{\sqrt{4-x^2}} \Rightarrow 4 - x^2 = x^2 \Rightarrow 4 = 2x^2$

$\Rightarrow x^2 = 2 \Rightarrow \boxed{x = \sqrt{2}}$ or $\cancel{\boxed{x = -\sqrt{2}}}$. So, test $x = 2$, $x = \sqrt{2}$, $x = -1$:

↑ Not in interval

$f(2) = 2\sqrt{4-(2)^2} = 2 \cdot 0 = 0$

$f(\sqrt{2}) = \sqrt{2}\sqrt{4-(\sqrt{2})^2} = \sqrt{2} \cdot \sqrt{2} = \boxed{2}$ ← Absolute max

$f(-1) = -1\sqrt{4-(-1)^2} = \boxed{-\sqrt{3}}$ ← Absolute min

10. (23 points) Consider the function $f(x) = x^4 - 4x^3 + 16$.

(a) Determine the critical values of $f(x)$ and find all open intervals of increase and all open intervals of decrease. Find the local maximum and minimum values.

(b) Find all open intervals on which the graph is concave up and all open intervals on which the graph is concave down, and determine the inflection points.

(c) Sketch the graph of $y = f(x)$ by hand, plotting and labeling **only** the relative extreme points, inflection points and the y-intercept. To be considered correct, your graph must match your answers in parts a) and b).

$$f'(x) = 4x^3 - 12x^2 = 0 \Rightarrow 4x^2(x-3) = 0 \Rightarrow 4x^2 = 0 \text{ or } x-3=0 \Rightarrow \boxed{x=0}, \boxed{x=3}$$

(a) Sign Chart:

$$\begin{array}{cccccc} f'(x) & \ominus & 0 & \ominus & 0 & \oplus \\ \hline x & \uparrow & 0 & \uparrow & 3 & \uparrow \\ & -1 & & 1 & & 100 \end{array}$$

f(x) is <u>increasing</u> on $[3, \infty)$

f(x) is <u>decreasing</u> on $(-\infty, 0) \cup (0, 3)$

Local min @ $\boxed{x=3}$

Local max DNE

(b) $$f''(x) = 12x^2 - 24x = 0 \Rightarrow 12x(x-2) = 0 \Rightarrow 12x = 0 \text{ or } x-2=0 \Rightarrow \boxed{x=0}, \boxed{x=2}$$

Sign Chart:

$$\begin{array}{cccccc} f''(x) & \oplus & 0 & \ominus & 0 & \oplus \\ \hline x & \uparrow & 0 & \uparrow & 2 & \uparrow \\ & -1 & & 1 & & 3 \end{array}$$

f(x) is <u>concave up</u> on $(-\infty, 0) \cup (2, \infty)$

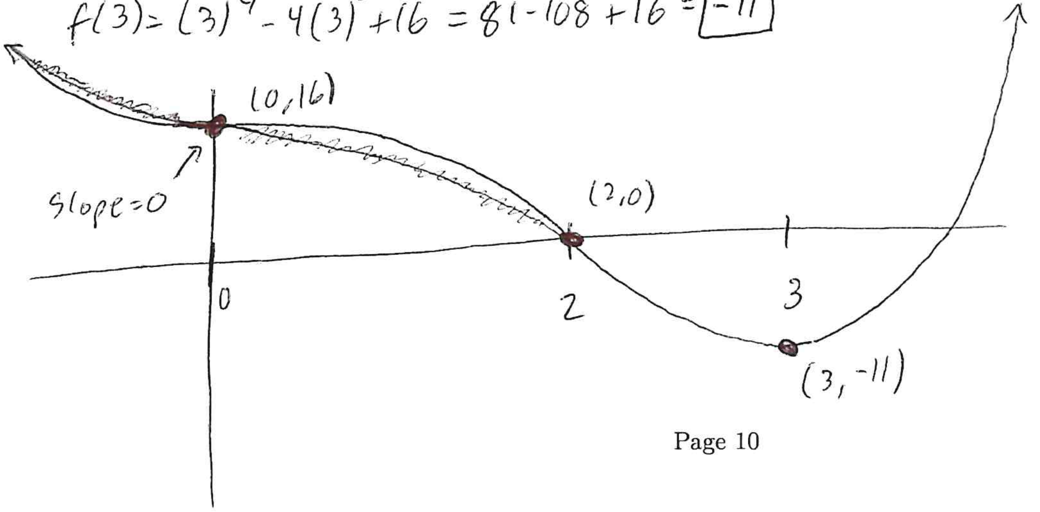
f(x) is <u>concave down</u> on $(0, 2)$

Inflection points @ $\boxed{x=0}$, $\boxed{x=2}$

(c) y-intercept at $f(0) = (0)^4 - 4(0)^3 + 16 = \boxed{16}$

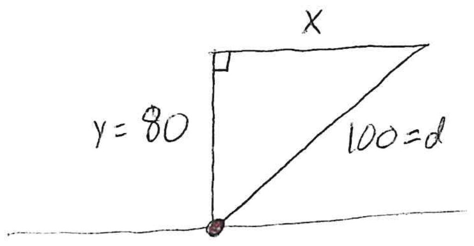
$f(0) = 16$, $f(2) = (2)^4 - 4(2)^3 + 16 = 16 - 32 + 16 = \boxed{0}$

$f(3) = (3)^4 - 4(3)^3 + 16 = 81 - 108 + 16 = \boxed{-11}$



11. (10 points) A kite 80 ft above the ground moves horizontally at a speed of 4 ft/s. At what rate is the string being released when 100 ft of string has been let out? Indicate clearly the units.

[illegible]



$x^2 + y^2 = d^2$ Differentiate w.r.t. time:

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2d\frac{dd}{dt}$. Now, $x = \sqrt{100^2 - 80^2} = \sqrt{3600} = 60$,

and $\frac{dx}{dt} = 4$, $\frac{dy}{dt} = 0$. Thus,

$2(60) \cdot 4 + 2 \cdot 80 \cdot 0 = 2 \cdot 100 \cdot \frac{dd}{dt}$

$$\frac{dd}{dt} = \frac{+480}{200} = \boxed{\frac{+12}{5}\text{ ft/sec}}$$

12. (12 points) A particle moves along a straight line with velocity $v(t) = 3t^2 - 6t$ (measured in meters per second).

(a) Find the displacement of the particle during the time period $1 \leq t \leq 3$.

(b) Find the total distance traveled by the particle during the same time period.

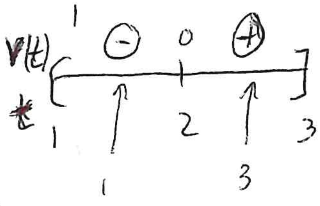
Displacement = Net change in distance = Final position - initial position = $\int_1^3 v(t)\,dt$

(a) $= \int_1^3 (3t^2 - 6t)\,dt = \left[t^3 - 3t^2\right]_1^3 = \left((3)^3 - 3(3)^2\right) - \left((1)^3 - 3(1)^2\right)$

$= 27 - 27 - (1 - 3) = \boxed{2}$

Total distance ⟺ All movement is positive, so compute $\int_1^3 |v(t)|\,dt = \int_1^3 |3t^2 - 6t|\,dt$.

[illegible] $v(t) = 0$ when $3t(t-2) = 0 \Rightarrow t = 0$ or $t = 2$. Sign chart:



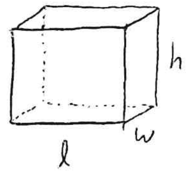
Get $\int_1^3 |3t^2 - 6t|\,dt = \int_1^2 (6t - 3t^2)\,dt + \int_2^3 (3t^2 - 6t)\,dt = \left[3t^2 - t^3\right]_1^2 + \left[t^3 - 3t^2\right]_2^3$

$= \left(3(2)^2 - (2)^3\right) - \left(3(1)^2 - (1)^3\right) + \left((3)^3 - 3(3)^2\right) - \left((2)^3 - 3(2)^2\right) = (12 - 8) - (3 - 1) + (27 - 27) - (8 - 12)$

$= 4 - 2 + 0 + 4 = \boxed{6}$

13. (13 points) A rectangular storage container with an open top is to have a volume of 10 m^3. The length of its base is twice the width. Material for the base costs \$5 per square meter. Material for the sides costs \$3 per square meter. Find the cost of materials for the cheapest such container and determine its dimension. Indicate clearly the units. You need to verify that your answer does give the lowest cost.

Objective: Minimize Cost $= 5 \cdot lw + 3 \cdot (2lh + 2wh)$



Constraint: Volume $V = 10 = lwh = 2w^2h \Rightarrow h = \frac{10}{2w^2} = \frac{5}{w^2}$

Constraint: $l = 2w$

$$\text{So, } C(w) = 5(2w)w + 3 \cdot \left(2(2w)\frac{5}{w^2} + 2w \cdot \frac{5}{w^2}\right)$$

$$= 10w^2 + \frac{60}{w} + \frac{30}{w} = 10w^2 + \frac{90}{w}$$

$$C'(w) = 20w - \frac{90}{w^2} = 0 \Rightarrow 20w = \frac{90}{w^2} \Rightarrow w^3 = \frac{9}{2} \Rightarrow \boxed{w = \sqrt[3]{\frac{9}{2}}}$$

Verify: $C''(w) = 20 + \frac{180}{w^3}$. When $w = \sqrt[3]{\frac{9}{2}}$, $C''(w) > 0$, so we have achieved a minimum as desired!

Thus, $w = \sqrt[3]{\frac{9}{2}}$, $l = 2 \cdot \sqrt[3]{\frac{9}{2}}$, $h = \frac{5}{\left(\sqrt[3]{\frac{9}{2}}\right)^2}$, and

$$\boxed{C = 5 \cdot 2 \cdot \left(\sqrt[3]{\frac{9}{2}}\right)^2 + 3 \cdot \left[2\left(2 \cdot \sqrt[3]{\frac{9}{2}}\right) \cdot \left(\frac{5}{\left(\sqrt[3]{\frac{9}{2}}\right)^2}\right) + 2\sqrt[3]{\frac{9}{2}} \cdot \frac{5}{\left(\sqrt[3]{\frac{9}{2}}\right.} \quad \text{dollars.}}$$